A note on rate-distortion functions for nonstationary Gaussian autoregressive processes

Robert M. Gray and Takeshi Hashimoto *

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Abstract

Source coding theorems and Shannon rate-distortion functions were studied for the discrete-time Wiener process by Berger and generalized to nonstationary Gaussian autoregressive processes by Gray and by Hashimoto and Arimoto. Hashimoto and Arimoto provided an example apparently contradicting the methods used in Gray, implied that Gray's rate-distortion evaluation was not correct in the nonstationary case, and derived a new formula that agreed with previous results for the stationary case and held in the nonstationary case. In this correspondence it is shown that the rate-distortion formulas of Gray and Hashimoto and Arimoto are in fact consistent and that the example of Hashimoto and Arimoto does not form a counterexample to the methods or results of the earlier paper. Their results do provide an alternative, but equivalent, formula for the rate-distortion function in the nonstationary case and they provide a concrete example that the classic Kolmogorov formula differs from the autoregressive formula when the autoregressive source is not stationary. Some observations are offered on the equality of the asymptotic distributions of the eigenvalues of the sequence of inverse autocorrelation matrices of possibly nonstationary autoregressive processes and of their Toeplitz approximations.
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1 Introduction

A real-valued Gaussian autoregressive source is defined by the difference equation

$$X_n = \begin{cases} -\sum_{k=1}^{n} a_k X_{n-k} + Z_n, & n = 1, 2, \cdots \\ 0, & n \le 0 \end{cases} \tag{1}$$

where the Z_n are i.i.d. (independently and identically distributed) random variables with mean zero and variance σ^2 and a_k are real numbers satisfying

$$\sum_{k=0}^{\infty} |a_k| < \infty \tag{2}$$

*Robert M. Gray is with the Information Systems Laboratory, Electrical Engineering Dept., Stanford University, Stanford, CA 94305. Takeshi Hashimoto is with the Dept. of Electronics Eng., University of Electro-Communications, 1-5-1 Chofugaoka, Chofushi, Tokyo 182-8585, Japan. This work was partially supported by the National Science Foundation under NSF Grant CCR-0309701 and by Hewlett-Packard Laboratories.

and $a_0 = 1$. The autoregressive process can be considered as the output of a linear filter described by a transfer function $1/A(z)$ driven by the memoryless process Z_n, where

$$A(z) = \sum_{k=0}^{\infty} a_k z^{-k}. \tag{3}$$

If the zeros of $A(z)$ (and hence the poles of the transfer function) all lie strictly inside of the unit circle, then the statistics of the autoregressive process approach a stationary distribution and the Shannon rate-distortion function of the process is given parametrically in $\theta \in (0, \infty)$ by Kolmogorov's classic formula [8] (see also [3]):

$$\begin{aligned} D(\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \min\left[\theta,\ \frac{\sigma^2}{g(\omega)}\right] d\omega && (4) \\ R(\theta) &= R_{\mathrm{K}}(\theta) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} \max\left[\frac{1}{2} \ln \frac{\sigma^2}{\theta g(\omega)},\ 0\right] d\omega && (5) \\ g(\omega) &= |A(e^{-j\omega})|^2 = \left| \sum_{k=0}^{\infty} a_k e^{-jk\omega} \right|^2 && (6) \end{aligned}$$

where the integral expression (5) is denoted by $R_K(\theta)$ as it will take a different form in the nonstationary case while the formula for distortion will remain the same.

Berger [2] proved a source coding theorem for the special case of a nonstationary autoregressive process with $a_1 = 1$ and $a_k = 0$ for $k > 1$ and he showed that the Kolmogorov formula still provided the rate-distortion function in this case. Gray [4] subsequently proved a source coding theorem for the general case described previously and derived a rate-distortion function resembling the Kolmogorov formula, but with (5) replaced by Eq. (22b) from [4] below:

$$R(\theta) = R_{\mathrm{AR}}(\theta) \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \ln\left(\max\left[g(\omega),\ \frac{\sigma^2}{\theta}\right]\right) d\omega. \tag{7}$$

Note that while (7) resembles the Kolmogorov form (5), it is not the same. Both forms are derived from the finite dimensional versions of the Kolmogorov formula, that is, the finite order rate-distortion functions. But the mechanics of taking the limit of the finite order expressions differ in the two cases in a critical way as will be described in detail later. The equivalence of the two formulas follows in the stationary case because of the existence of source coding theorems for each, but it does not follow in the nonstationary case.

The two rate expressions $R_{\mathrm{AR}}(\theta)$ and $R_{\mathrm{K}}(\theta)$ can be related to each other as follows. Define the subset $E = \{\omega : g(\omega) < \sigma^2/\theta\}$ of $[-\pi, \pi]$. Then

$$\begin{aligned} R_{\mathrm{AR}}(\theta) - R_{\mathrm{K}}(\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} \ln\left(\max\left[g(\omega), \frac{\sigma^2}{\theta}\right]\right) d\omega - \frac{1}{2\pi} \int_{-\pi}^{\pi} \max\left[\frac{1}{2} \ln \frac{\sigma^2}{\theta g(\omega)}, 0\right] d\omega \\ &= \frac{1}{2\pi} \int_{E^c} \left(\frac{1}{2} \ln g(\omega)\right) d\omega + \left(\frac{1}{2} \ln \frac{\sigma^2}{\theta}\right) \frac{1}{2\pi} \int_{E} d\omega - \frac{1}{2\pi} \int_{E} \left[\frac{1}{2} \ln\left(\frac{\sigma^2}{\theta g(\omega)}\right)\right] d\omega \\ &= \frac{1}{2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln g(\omega)\, d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |A(e^{-j\omega})|\, d\omega && (8) \end{aligned}$$

and hence the two formulas will agree if and only if the final integral is 0.

In 1980 Hashimoto and Arimoto [7] revisited the question of the rate-distortion function in the nonstationary case. They considered the finite order autoregressive case and noted that both the source coding theorem and the evaluation of the rate-distortion function had been accomplished for the Wiener process in [2], but they only described the source coding theorem and not the rate distortion function of [4] for the more general autoregressive case, stating that "the rate-distortion function has not been calculated for nonstationary processes except for the Wiener process" and presented an "example which shows the form (3) is incorrect if the process is not asymptotically stationary, and we present the exact form of the rate-distortion in the next section." Their equation (3), however, corresponds to the Kolmogorov form $R_{\mathrm{K}}(\theta)$ of (5) and not the autoregressive form $R_{\mathrm{AR}}(\theta)$ of (7), so that their example provided a demonstration that the Kolmogorov formula fails in the nonstationary case, but not that there was a problem with the autoregressive result (7) of [4]. As a result, there has been some confusion about the validity of the rate-distortion function of [4] in the nonstationary case and the apparently different result provided in [7] as well as some confusion about applicability of the specific asymptotic eigenvalue results for Toeplitz matrices used in [4].

We here reconcile the two forms of $R_{\mathrm{AR}}(\theta)$ and the formula of Hashimoto and Arimoto for the nonstationary case and demonstrate that they are equal and distinct from the Kolmogorov formula for nonstationary autoregressive processes. We also remark on some related issues regarding the eigenvalue distributions of sequences of inverse autocorrelation matrices of autoregressive processes and of their natural Toeplitz approximations.

2 Nonstationary autoregressive processes revisited

For the Mth-order autoregressive process ($a_k = 0$ for $k > M$), Hashimoto and Arimoto correctly point out that the Kolmogorov formula (5) (their (3)) fails for a simple first order nonstationary autoregressive source and they state their main result, which replaces (5) in the Kolmogorov formula by the form

$$R(\theta) = R_{\mathrm{HA}}(\theta) \triangleq R_{\mathrm{K}}(\theta) + \sum_{k=1}^{M} \max\left[\frac{1}{2}\ln|\rho_k|^2,\ 0\right] \tag{9}$$

where ρ_k are the zeros of $A(z) = \sum_{k=0}^{M} a_k z^{-k}$. Suppose that $|\rho_1| \geq \cdots \geq |\rho_m| > 1 > |\rho_{m+1}| \geq \cdots \geq |\rho_M|$. Then, (9) can be rewritten as

$$R_{\mathrm{HA}}(\theta) = R_{\mathrm{K}}(\theta) + \sum_{k=1}^{m} \frac{1}{2}\ln|\rho_k|^2. \tag{10}$$

In the stationary case, there are no zeros outside of the unit circle and R_{HA} reduces to R_{K}. R_{HA} can be related to the autoregressive formula $R_{\mathrm{AR}}(\theta)$ by means of Jensen's formula (or the Jacobi-Jensen formula; see, e.g., [1] or [9]). In fact, the relationship holds more generally for infinite order autoregressive processes if (2) and $a_0 = 1$ hold and hence the more general case is considered.

Define

$$f(z) = A(1/z) = \sum_{k=0}^{\infty} a_k z^k$$

and observe that from (7)

$$
\begin{aligned}
R_{\mathrm{AR}}(\theta) - R_{\mathrm{K}}(\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|A(e^{-j\omega})|\, d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|f(e^{j\omega})|\, d\omega. \qquad (11)
\end{aligned}
$$

The function $f(z)$ is analytic in a region containing the closed unit circle and $f(0) = a_0 = 1$ and hence it must contain only a finite number of zeros, say α_i, $i = 1, \ldots, \tilde{m}$, inside the unit circle with multiple zeros repeated. This follows from the properties of analytic functions since the presence of an infinite number of zeros within the unit circle would imply the existence of an accumulation point, which would force f to be identically zero within the unit circle, which would contradict the assumption $f(0) = 1$. Thus Jensen's formula can be applied to write

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|f(e^{j\omega})|\, d\omega = \sum_{i=1}^{\tilde{m}} \ln\frac{1}{|\alpha_i|}. \qquad (12)
$$

The zeros of $f(z)$ inside the unit circle are the reciprocals of the zeros of $A(z)$ outside the unit circle, that is, $\tilde{m} = m$ and $\alpha_i = 1/\rho_i$, $i = 1, \ldots, m$. Furthermore, as discussed in [1] and [9], Jensen's formula remains true for zeros on the unit circle as well as within the unit circle and hence, for any autoregressive process described by $A(z)$ satisfying (2) and $a_0 = 1$, it results in

$$
R_{\mathrm{AR}}(\theta) = R_{\mathrm{K}}(\theta) + \sum_{k=1}^{m} \ln|\rho_k|, \qquad (13)
$$

where $\rho_1, \cdots, \rho_m$ are the zeros of $A(z)$ outside or on the unit circle. This formula agrees with the rate-distortion function of (10) in the finite-order autoregressive case and generalizes that result to the infinite order case with absolutely summable coefficients. Thus the results of [7] demonstrate that the Kolmogorov formula may fail for nonstationary sources, not that the autoregressive formula is incorrect. The two formulas agree for stationary sources and for the nonstationary Wiener process and more generally for all nonstationary autoregressive processes satisfying (7). Note that (13) has the interpretation that the correction term needed for the Kolmogorov formula in the nonstationary case is the sum of the log moduli of the unstable poles.

3 Asymptotic eigenvalue distributions

Although the rate-distortion functions of [4] and [7] are equivalent, their derivations apply the classic asymptotic eigenvalue distribution theorem for Toeplitz matrices in two different ways. The classic form can be described as follows. Given a discrete-time Fourier transform pair

$$
\begin{aligned}
f(\omega) &= \sum_{k=-\infty}^{\infty} t_k e^{-jk\omega}; \omega \in [-\pi, \pi) \qquad (14) \\
t_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\omega) e^{jk\omega}\, d\omega; k = \ldots, -1, 0, 1, \ldots \qquad (15)
\end{aligned}
$$

let $T_n = T_n(f(\omega)) \triangleq \{t_{k-\ell}; k, \ell = 0, 1, \ldots, n-1\}$ be the corresponding Toeplitz matrix with eigenvalues $\tau_{n,k}, k = 0, 1, \ldots, n-1$. Denote the essential infimum and supremum of f by m_f and M_f, respectively. Then the classical theorem (see, e.g., Section 7.4 of [6] or the tutorial [5]) states that, if F is a continuous function on $[m_f, M_f]$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} F(\tau_{n,k}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F[f(\omega)]\, d\omega. \tag{16}$$

The theorem extends to matrices which are not necessarily Toeplitz matrices. Given an $n \times n$ matrix $B = \{b_{k,\ell}; k, \ell = 0, 1, \ldots, n-1\}$ with eigenvalues λ_k, $k = 0, 1, \ldots, n-1$, let its strong and weak norms be, respectively,

$$\|B\| = \max_k |\lambda_k| \;\text{ and }\; |B| = \left\{ \frac{1}{n} \sum_{k,\ell} |B_{k,\ell}|^2 \right\}^{\frac{1}{2}}$$

It is known (see the above reference) that, if a sequence of $n \times n$ matrices B_n is uniformly bounded in both norms and satisfies

$$\lim_{n\to\infty} |B_n - T_n| = 0 \tag{17}$$

then (16) will also hold for the eigenvalues of B_n; that is, if $\lambda_{n,k}, k = 0, 1, \ldots, n-1$ are the eigenvalues of B_n, then (16) holds with $\tau_{n,k}$ replaced by $\lambda_{n,k}$. Such a sequence of matrices B_n is said to be *asymptotically equivalent* to the sequence T_n. Two sequences of eigenvalues $\tau_{n,k}$ and $\lambda_{n,k}$ constrained to a common finite region $[m, M]$ are said to be *asymptotically equally distributed* or to have equal asymptotic distributions if the equality

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} F(\tau_{n,k}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} F(\lambda_{n,k})$$

holds for all F continuous on $[m, M]$. Thus sequences of asymptotically equivalent matrices will have asymptotically equally distributed eigenvalues.

The classic Kolmogorov result for a stationary autoregressive Gaussian process follows from his finite order results by taking B_n and $\lambda_{n,k}$ as the nth order covariance matrix of the Gaussian process and the corresponding eigenvalues. The limit is computed by demonstrating the asymptotic equivalence of B_n and a Toeplitz approximation T_n with eigenvalues $\tau_{n,k}$ and using the asymptotic equivalence of eigenvalues to compute the limit

$$D(\theta) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \min(\theta,\ \tau_{n,k}) \tag{18}$$

$$R_{\mathrm{K}}(\theta) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \max\left(0,\ \frac{1}{2} \ln \frac{\tau_{n,k}}{\theta}\right). \tag{19}$$

The autoregressive result instead focuses on the inverse covariance. The difference equation defining an autoregressive process can be written in vector form as

$$A_n X^n = Z^n.$$

where the lower triangular Toeplitz matrix A_n is given by

$$A_n = \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ a_1 & 1 & 0 & \cdots & & 0 \\ & a_1 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & \\ a_{n-2} & \cdots & & & & 0 \\ a_{n-1} & & \cdots & & a_1 & 1 \end{bmatrix}. \tag{20}$$

The inverse covariance

$$\Phi_n = \sigma^{-2} A_n^* A_n \tag{21}$$

is then asymptotically equivalent to the Toeplitz matrix $T_n(g(\omega)/\sigma^2)$ determined from the inverse Fourier transform of $g(\omega)/\sigma^2$ with $g(\omega)$ in (6) and hence the Toeplitz eigenvalue distribution theorem can be applied with $\tau_{n,k} = 1/\lambda_{n.k}$, where now the $\lambda_{n,k}$ are the eigenvalues of $\Psi_n = \sigma^{-2} A_n^* A_n$.

As Hashimoto and Arimoto point out, in the nonstationary case direct application of the asymptotic eigenvalue distribution theorem does not work in evaluating the limit of (19) because of the behavior of the $\lambda_{n,k}$ near zero. Furthermore, they state that in this case that "the eigenvalues of Ψ_n and Φ_n have distinct distributions unless the process is asymptotically stationary." Alternatively, the failure of the classic asymptotic eigenvalue theorem for sequences of Toeplitz matrices to apply is a direct result of the fact that the required conditions of bounded eigenvalues of the autocorrelation matrix are violated for nonstationary autoregressive processes. This difficulty is obvious from rewriting (19) as

$$R(\theta) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \max\left(0,\ \frac{1}{2} \ln \frac{1}{\lambda_{n,k}\theta}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} G(\lambda_{k,n}) \tag{22}$$

where

$$G(\lambda) = \max\left(0,\ \frac{1}{2} \ln \frac{1}{\lambda\theta}\right) \tag{23}$$

is not continuous at $\lambda = 0$. Hashimoto and Arimoto circumvent this difficulty by the observation that exactly the m smallest $\lambda_{n,k}$ decrease exponentially as n increases while the remaining $\lambda_{n,k}$ are bounded from zero. Between those m smallest $\lambda_{n,k}$, the ℓth smallest one decreases asymptotically as $|\rho_\ell|^{-2n}$, for $\ell = 1,\ 2,\ \cdots\ ,\ m$, and the expression (9) follows.

The derivation of [4], however, avoided the above difficulty by deriving an equivalent form to the Kolmogorov finite order formula (see (15b)–(17b) of [4]) :

$$\frac{1}{n} \sum_{k=0}^{n-1} \max\left(0,\ \frac{1}{2} \ln \frac{1}{\lambda_{n,k}\theta}\right) = \frac{1}{n} \sum_{k=0}^{n-1} \ln\left[\max\left(\sigma^2\lambda_{n,k},\ \frac{\sigma^2}{\theta}\right)\right] = \frac{1}{n} \sum_{k=0}^{n-1} F(\lambda_{n,k}) \tag{24}$$

where

$$F(\lambda) = \ln\left[\max\left(\sigma^2\lambda,\ \frac{\sigma^2}{\theta}\right)\right] \tag{25}$$

is continuous at $\lambda = 0$ and hence the limit can be evaluated immediately by the classical Toeplitz eigenvalue theorem. This yields an answer with a different functional form from the traditional Kolmogorov formula which is not contradicted by the example of [7] and which has no problems with $\lambda_{n,k}$ near 0.

The previously quoted statement in Hashimoto and Arimoto [7] that the asymptotic eigenvalue distributions of the matrices $\Phi_n = \sigma^{-2}A_n^* A_n$ and $\Psi_n = T_n(g(\omega)/\sigma^2)$ are distinct in the nonstationary case where the essential infimum of $g(\omega) = 0$ was based on the demonstrated failure of the limit of (22) to equal $R_K(\theta)$ in this case. This failure does not demonstrate that Φ_n and Ψ_n are not asymptotically equivalent in the sense of being bounded and satisfying (17), however, since the $G(\lambda)$ of (23) is not continuous at $\lambda = 0$ and hence does not provide a counterexample to the implications of asymptotically equivalent eigenvalue distributions. In fact, it is shown in [4] that the two sequences of matrices *are* asymptotically equivalent and therefore have the same asymptotic eigenvalue distributions and hence the autoregressive form of the rate-distortion function follows by direct application of the asymptotic equivalence and the classical Toeplitz asymptotic eigenvalue distribution theorem.

What is true in the nonstationary case is that the asymptotic distributions of the eigenvalues $\tau_{n,k}$ of the autocorrelation matrices Φ_n^{-1} and those of $T_n(\sigma^2/g)$ are not the same for the simple reason that these eigenvalue sequences are not bounded. As a result the usual Kolmogorov formulation, which is in terms of the eigenvalues of the correlation matrices Φ_n^{-1}, does not yield a solution by direct application of the Toeplitz asymptotic eigenvalue theorem. The reformulation in terms of the inverse correlation matrix eigenvalues provides an example of where such a limit can be evaluated by taking advantage of the asymptotic equivalence of the corresponding inverse matrices.

4 Conclusion

The rate-distortion formulas of [4] and [7] are consistent and the results of the latter provide no evidence of invalidity of the former. The two papers provide alternative characterizations of the same quantity which are related through the Jacobi-Jensen formula. The second paper provided the first detailed example where the Kolmogorov and autoregressive formulas for the rate-distortion function differed by a nonzero amount. Contrary to the statement in [7], the asymptotic sequence eigenvalue distributions of the sequences of matrices $\Phi_n = \sigma^{-2}A_n^* A_n$ and $\Psi_n = T_n(g(\omega)/\sigma^2)$ are identical in both the stationary and nonstationary cases, but the corresponding inverses have asymptotically equally distributed eigenvalues only in the stationary case.

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