TRANSFER FUNCTION BOUNDS ON PERFORMANCE OF TURBO CODES FOR M-ARY ORTHOGONAL SIGNALS

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Abstract — We obtain upper bounds on the bit-error rate (BER) for maximum-likelihood (ML) decoding of turbo codes constructed with uniform interleavers for M-ary orthogonal signals. We use transfer function bounding techniques to obtain the above bounds. Thus, the upper bounds result in the average bound over all interleavers of a given length. We apply the techniques to parallel concatenated coding (PCC) schemes where recursive convolutional codes are used as constituent codes. We present the average bounds on the BER of turbo codes with constraint length 3 for M-ary orthogonal signals on additive white Gaussian noise (AWGN) channels. We show that the performance advantage of turbo-coded 4-ary orthogonal signals over turbo-coded binary orthogonal signals is smaller than the performance advantage in the absence of coding, which is 3.0 dB.

I. Introduction

Recently, turbo codes have attracted much attention because of their astonishing error performance with their reasonable decoding complexity [1]-[4]. Turbo codes were originally proposed for binary modulation. To improve bandwidth efficiency, turbo codes have been applied to multi-level/phase modulations, e.g., 8-PSK, 16-QAM, and 64-QAM [5]-[8]. In these studies, the code performance was evaluated via computer simulations. References [6] and [7] show that for a given bandwidth efficiency and interleaver length turbo-coded modulation schemes have performance superior to all known schemes, such as trellis-coded modulation employing Ungerboeck codes.

In addition to computer simulations, it is also very useful to have theoretical bounds in the range where the computer simulations is not feasible. Some researchers have developed the very useful transfer function technique to obtain upper bounds on the BER for maximum-likelihood (ML) decoding of turbo codes for binary orthogonal signals [2]-[4]. Note that there are some differences between the transfer-function bounds for turbo codes and those for convolutional codes [2]. First, the bounds for turbo codes require a joint enumerator for all possible combinations of input weights and output weights of error events. Second, since turbo codes are block codes, it is difficult to enumerate compound error events that include more than one excursion from the all-zero state within the fixed block length. Third, since it is difficult to obtain the bound for any particular choice of permutation, the bound is developed as an average over randomly chosen interleavers. Finally, the bounds are derived as upper bounds on the BER of a ML decoder. These bounds are known to be useful for evaluating the error performance where the computer simulations are not feasible. However, the bounds on the error probability of turbo codes for M-ary orthogonal signals have not been developed.

In this paper, we use the transfer function technique to obtain upper bounds on the BER for ML decoding of turbo codes constructed with uniform interleavers for M-ary orthogonal signals. Thus, our result corresponds to the average upper bound over all interleavers of a given length. We apply these techniques to parallel concatenated coding (PCC) schemes where recursive convolutional codes are used as constituent codes. We present the average bounds on the BER of turbo coded M-ary orthogonal signals with constraint length 3 on additive white Gaussian noise (AWGN) channels. We show that the performance advantage of turbo-coded 4-ary orthogonal signals over turbo-coded binary orthogonal signals is smaller than the performance advantage in the absence of coding, which is 3.0 dB.

II. Turbo Code Examples for 4-ary Orthogonal Signals

We consider the two particular turbo codes presented in [2][4]: (1, 7/5, 7/5) and (1, 5/7, 5/7). The parity streams in these encoders come from the simple recur-

sive convolutional encoders with constraint length ν=3. The three separate rate-1 components of the code are referred to as code fragments. Each code fragment is preceded by a permutation. A full codeword of the turbo code depends only on the relative permutation between the inputs.

To analyze the performance of the turbo codes theoretically, we want to know, for a given interleaver, the input-output weight enumerator function (IOWEF), sometimes called the input-redundancy weight enumerating function (IRWEF.) However, to obtain the IOWEF for a given interleaver is exceedingly complicated, because the redundant bits generated by the second constituent encoder will depend not only on the weight of the input word, but also on the permutation preceding the encoder. To overcome this difficulty, we introduce a random interleaver, sometimes called, a uniform interleaver [2][3]. A random interleaver of length N is a probabilistic device that maps a given input word of weight d into all its $\binom{N}{d}$ distinct permutations, each with equal probability $1/\binom{N}{d}$. Each of the encoders is used to generate a $(3(N+2), N)$ block code, where N is the information block length. Following the information bits, an additional 2 "tail bits" are appended to initialize the encoder, that is, to drive the encoder to the all-zero state at the end of the block. Here we use the termination method described in [9]. Since the component encoders are recursive, it is impossible to terminate both component encoders with the same m bits. Usually only the first encoder is forced to to return to the initial state by appending tail bits. Note that the performance degradation caused by the ambiguity of the final state of the second encoder is negligible for a large N.

We consider the state transitions for M-ary orthogonal signals. There are various ways to map the output encoded bits to the signals. When we map $\log_2 M$ encoded bits selected from several different code fragments to the same M-ary signal, it is exceedingly complicated to derive the state transition matrix, because we have to consider the various combinations of code fragments, that is, we need to consider the hyper trellis. To overcome this difficulty, we map $\log_2 M$ encoded bits from one code fragment to each M-ary signal. For 4-ary orthogonal signals, each pair of output bits from a code fragment specifies a 4-ary orthogonal symbol.

To analyze the performance of the turbo-coded M-ary orthogonal signals, we need to consider the number of transmitted symbols by which a correct path and a wrong path differ. We introduce a monomial $L^l I^i K^k$. For M-ary orthogonal signals, l describes the number of output bits per transmitted signal, i.e., $l = \log_2 M$. The variable i describes the weight of input bits per transmitted symbol, i.e., $i = 0, 1, \cdots, \log_2 M$. The variable k is either 0 or 1, depending on whether or not the corresponding output bits differ from an all-zero block of length $\log_2 M$. In the case of 4-ary orthogonal signals, the state transition matrices are given by

$$\mathbf{A}^2_{7/5}(L,I,D) = \begin{pmatrix} L^2 & L^2IK & L^2IK & L^2I^2K \\ L^2IK & L^2K & L^2I^2 & L^2IK \\ L^2IK & L^2I^2K & L^2K & L^2I \\ L^2I^2K & L^2I & L^2IK & L^2K \end{pmatrix} \tag{1}$$

for the (7/5) code fragment, and

$$\mathbf{A}^2_{5/7}(L,I,D) = \begin{pmatrix} L^2 & L^2IK & L^2I^2K & L^2IK \\ L^2I^2K & L^2I & L^2K & L^2IK \\ L^2IK & L^2I^2K & L^2I & L^2K \\ L^2IK & L^2K & L^2IK & L^2I^2 \end{pmatrix} \tag{2}$$

for the (5/7) code fragment. Note that each transition is labeled with L^2, because two bits are needed for each transition for 4-ary orthogonal signals using the mapping in this paper.

We derive the IOWEF for a given code fragment with 2^m states, where m is the number of shift registers used to generate the code fragment. The variable $t(l,i,k)$ denotes the number of paths of length l, input weight i, and having k output symbols that differ from the all-zero symbol, where these paths start and end in state 0^m, with the proviso that the last m edges in the path are "termination" edges and have neither length nor input weight. Following [2], the corresponding transfer function (generating function) is defined by

$$T(L,I,K) = \sum_{l\geq 0}\sum_{i\geq 0}\sum_{k\geq 0} L^l I^i K^k t(l,i,k). \tag{3}$$

Following [2], for 4-ary orthogonal signals, $T(L,I,K)$ is the $(0^m, 0^m)$ entry in the matrix

$$\begin{aligned}&(\mathbf{I} + \mathbf{A}^2(L,I,K) + \mathbf{A}^2(L,I,K)^2 + \mathbf{A}^2(L,I,K)^3 \\ &+\mathbf{A}^2(L,I,K)^4 + \cdots)\mathbf{A}(1,1,K)^m\end{aligned} \tag{4}$$

where 0^m represents an all-zero m-tuple. The factor $\mathbf{A}(1,1,K)$ corresponds to the termination edges. Since $\mathbf{I} + \mathbf{A}^2(L,I,K) + \mathbf{A}^2(L,I,K)^2 + \mathbf{A}^2(L,I,K)^3 + \cdots = (\mathbf{I} - \mathbf{A}^2)^{-1}$, the transfer function becomes:

$$T(L,I,K) = [(\mathbf{I} - \mathbf{A}^2(L,I,K))^{-1} \cdot \mathbf{A}(1,1,K)^m]_{0^m,0^m}. \tag{5}$$

Approximating (5) by omitting the termination factor $\mathbf{A}(1,1,K)^m$, the transfer functions for the (7/5) and (5/7) code fragments for 4-ary orthogonal signals can be derived.

III. Union Bounds on Word and Bit Error Probabilities

We derive a union bound on the probabilities of word error and bit error for turbo-coded M-ary orthogonal signals by using the enumerators $t(l,i,k)$ for each code fragment of the turbo code. We assume an AWGN channel with symbol energy-to-noise ratio E_s/N_0.

First we derive the IOWEF for the code fragment (1), that is, the uncoded branch. As noted previously, we introduce a random interleaver. Each code fragment of the turbo code is preceded by a random permutation of the input information bits $\mathbf{u}$. The random interleavers transform the input sequence $\mathbf{u}$ into three permuted sequence $\mathbf{u_0}$, $\mathbf{u_1}$, and $\mathbf{u_2}$. Each permuted sequence has the same Hamming weight as the original sequence. Since the turbo code can be viewed as a block code of length N, there are $t^2_{7/5}(N,i,k)$ codeword fragments of input weight i, and k is the number of output symbols different from the correct all-zero symbol among the two (7/5) codeword fragments of the (1,7/5,7/5) code for 4-ary orthogonal signals. Similarly, there are $t^2_{5/7}(N,i,k)$ codeword fragments of input weight i, and k is the number of output symbols different from the correct all-zero symbol among the two (5/7) code fragments of the (1,5/7,5/7) code for 4-ary orthogonal signals. The number of codeword fragments from the code fragment (1) for M-ary orthogonal signal is derived as

$$
\begin{aligned}
&t_1^{\log_2 M}(N,i,j) = \\
&\quad \binom{N/\log_2 M}{j} \sum_{i_1=i_1(N,i,j)_{min}}^{i_1(N,i,j)_{max}} \binom{\log_2 M}{i_1} \\
&\quad \cdot \sum_{i_2=i_2(N,i,j)_{min}}^{i_2(N,i,j)_{max}} \binom{\log_2 M}{i_2} \cdots \sum_{i_r=i_r(N,i,j)_{min}}^{i_r(N,i,j)_{max}} \binom{\log_2 M}{i_r} \\
&\quad \cdots \sum_{i_{N/\log_2 M-j}=i_{N/\log_2 M-j}(N,i,j)_{min}}^{i_{N/\log_2 M-j}(N,i,j)_{max}} \binom{\log_2 M}{i_{N/\log_2 M-j}}, \\
&\quad j \in (j = j_{min}(N,i) \cdots j_{max}(N,i))
\end{aligned} \tag{6}
$$

where j represents the number of correct symbols, i.e. equal to an all-zero tuple of length $\log_2 M$, and other variables are given by the following equations:

$$
\begin{aligned}
i_r(N,i,j)_{min} &= \max(1, i - \sum_{h=1}^{r-1} i_h - N + j\log_2 M \\
&\quad + r\log_2 M), \qquad (7) \\
i_r(N,i,j)_{max} &= \min(\log_2 M, i - N/\log_2 M + j + r \\
&\quad - \sum_{h=1}^{r-1} i_h), \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
j_{min}(N,i) &= \max\left\{0, \frac{N}{\log_2 M} - i\right\}, \qquad (9) \\
j_{max}(N,i) &= \left\lfloor \frac{N-i}{\log_2 M} \right\rfloor. \qquad (10)
\end{aligned}
$$

For 4-ary orthogonal signal, $t_1^2(N,i,j)$ can be simplified as follows.

$$
\begin{aligned}
t_1^2(N,i,j) &= \binom{N/2}{j}\binom{N/2-j}{i-N/2+j} \cdot 2^{N-2j-i}, \\
j &\in (j = j_{min}(N,i), \cdots, j_{max}(N,i)). \qquad (11)
\end{aligned}
$$

Denote by $p(k|i)$ the conditional probability of decoding a codeword fragment differing in k symbols from the correct one given a randomly selected input sequence of weight i. Following [2], $p(k|i)$ can be written as

$$
p(k|i) = \frac{t^{\log_2 M}(N,i,k)}{\sum_{k'} t(N,i,k')} = \frac{t^{\log_2 M}(N,i,k)}{\binom{N}{i}}. \tag{12}
$$

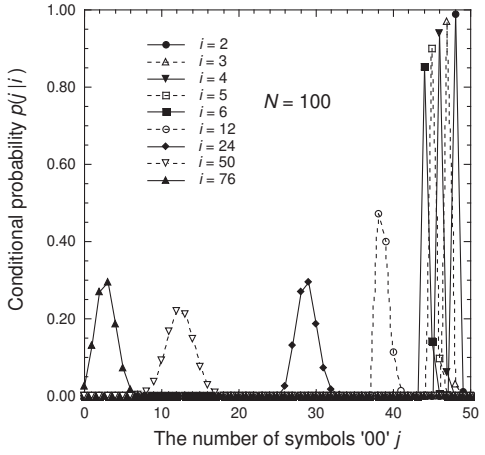
Figs. 1 (a), (b), and (c) show the conditional probability distributions $p_1(j|i)$, $p_{7/5}(k|i)$, and $p_{5/7}(k|i)$ for the three code fragments (1), (7/5), and (5/7) for the block length N=100 and M=4, respectively, i.e., for 4-ary orthogonal signals. From Fig. 1 (a), we can see that as the input weight increases, the value of j at which $p_1(j|i)$ is maximized becomes smaller. This is obvious from the definition of j as the number of correct symbols. In both Figs. 1 (b) and (c), the distribution of the conditional probability $p(k|i)$ approaches the binomial distribution for large values of the input weight i. This means that for large input weight i, the conditional probability $p(k|i)$ exhibits more randomness in the number of symbols differing from the correct one for 4-ary orthogonal signals. For small values of the input weight i, the conditional distributions $p(k|i)$'s are markedly different from the binomial distribution relevant to random (i.i.d.) codes. Note that, as for binary orthogonal signals, the vertical scale of Fig. 1 (b) is about twice that of Fig. 1 (c), because the (7/5) code fragment admits only even input weights i.

Given a particular selection of the random interleaver, i.e., the random permutations, the probability $\tilde{p}(j,k_1,k_2|i)$ that an input sequence $\mathbf{u}$ of weight i will be mapped into a M-ary orthogonal symbols differing in $N/\log_2 M - j + k_1 + k_2$ symbols from the correct sequence (a sequence of all-zero symbols) can be derived as

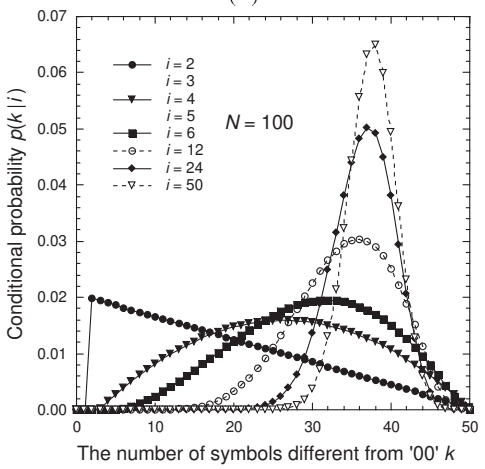
$$
\tilde{p}(j,k_1,k_2|i) = p_1(j|i)p_{7/5}(k_1|i)p_{7/5}(k_2|i) \tag{13}
$$

for the (1,7/5,7/5) code and

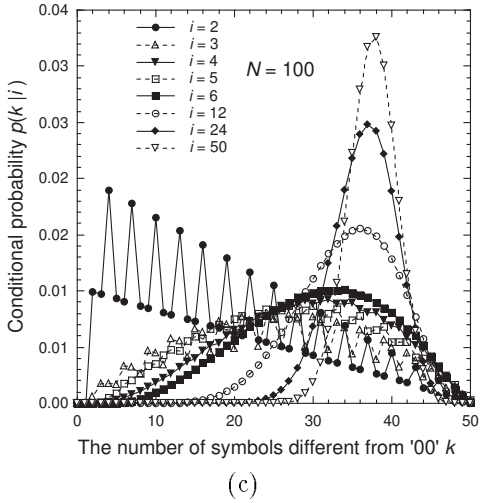
$$
\tilde{p}(j,k_1,k_2|i) = p_1(j|i)p_{5/7}(k_1|i)p_{5/7}(k_2|i) \tag{14}
$$



(a)



(b)



(c)

Fig. 1: Distributions of conditional probabilities $p(j|i)$ and $p(k|i)$ for the numbers of output symbols same as and different from '00' j and k, respectively, given input weight i: (a) $p(j|i)$ for the code fragment (1), (b) $p(k|i)$ for the code fragment (7/5), and (c) for the code fragment (5/7.)

for the (1,5/7,5/7) code. The conditional probability that a ML decoder selects a particular codeword of total weight $d = N/\log_2 M - j + k_1 + k_2$ instead of the all-zero codeword is $Q(\sqrt{dE_s/N_0})$, where $Q(\cdot)$ is a Gaussian Q-function. Thus, the information bit-error probability P_b is upper bounded by

$$P_b \leq \sum_{i=1}^{N} \frac{i}{N} \sum_{j=j_{min}(N,i)}^{j_{max}(N,i)} \sum_{k_1=0}^{N/\log_2 M} \sum_{k_2=0}^{N/\log_2 M} p_1(j|i)p(k_1|i)p(k_2|i)Q\left(\sqrt{\frac{dE_s}{N_0}}\right). \quad (15)$$

Note that the bit-error probability P_b bounded in (15) is an average over randomly chosen permutations [2].

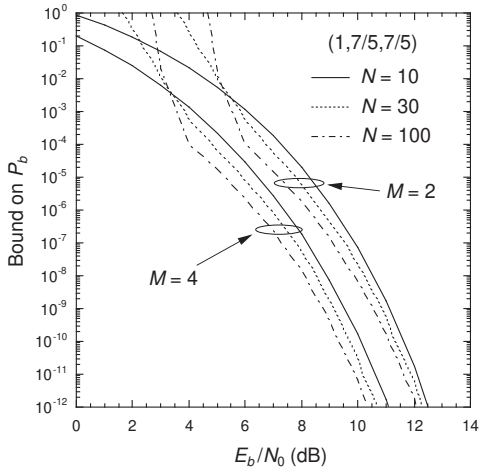
IV. Numerical Results

Figs. 2 (a) and (b) show the bounds on P_b for the (1,7/5,7/5) code and the (1,5/7,5/7) code with the various values of the block length N for binary and 4-ary orthogonal signals, respectively. As reported in [2]-[4], the bounds on P_b for binary orthogonal signals abruptly become useless roughly below a threshold value of the information bit energy-to-noise ratio E_b/N_0 determined by the computational cutoff rate R_0 if the block length N is large. We found that the bounds on P_b for 4-ary orthogonal signals exhibit this behavior. We also found that the performance advantage of turbo-coded 4-ary orthogonal signals over turbo-coded binary orthogonal signals is smaller than the performance advantage in the absence of coding, which is 3.0 dB.

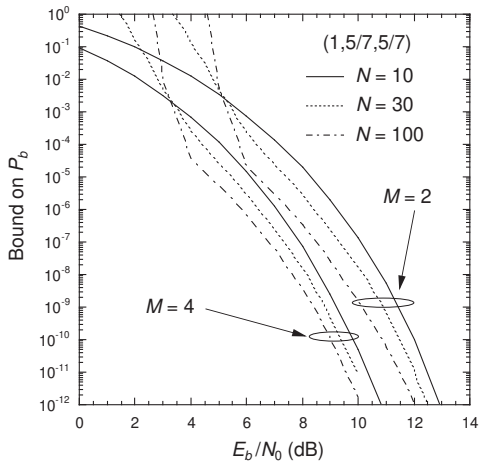
Fig. 3 compares the computed bounds for the (1,7/5,7/5) and (1,5/7,5/7) turbo-coded binary, 4-ary, and 8-ary orthogonal signals where the block length is N=30. From Fig. 5, we found that when M=8, the (1,5/7,5/7) code has better performance, while when M=2 and 4, both the codes have almost the same performance.

V. Conclusion

We have obtained upper bounds on the BER for ML decoding of turbo codes constructed with uniform interleavers for M-ary orthogonal signals. The bounds were derived using the transfer function bounding techniques. We applied the techniques to the PCCC schemes. We showed that, comparing the performance of binary orthogonal signals to that of 4-ary orthogonal signals, the performance advantage of turbo-coded 4-ary orthogonal signals over turbo-coded binary orthogonal signals is smaller than the performance advantage in the absence of coding, which is 3.0 dB. We



(a)



(b)

Fig. 2: Bounds on P_b for the (1,7/5,7/5) and (1,5/7,5/7) codes with various block length N for binary and 4-ary orthogonal signals: (a) the (1,7/5,7/5) code and (b) the (1,5/7,5/7) code.

also showed that when M=8, the (1,5/7,5/7) code has better performance, while when M=2 and 4, both the codes have almost the same performance.

Acknowledgements

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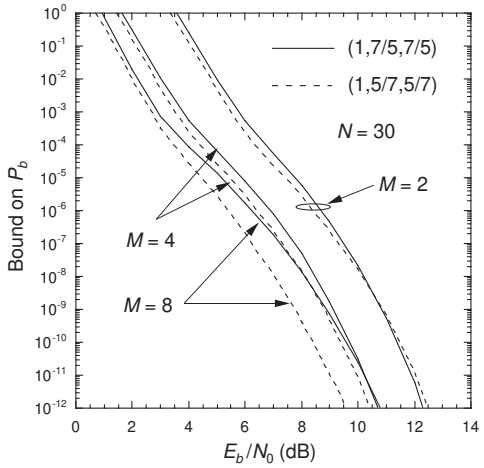

Fig. 3: Bounds on P_b for the (1,7/5,7/5) and (1,5/7,5/7) turbo code with N=30 for binary, 4-ary, and 8-ary orthogonal signals.

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