Performance of Turbo Codes with Two-Branch Receive Diversity and Correlated Fading

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Abstract — We analyze the effects of fading correlation on the performance of turbo codes with two-branch receive diversity combining schemes. We use transfer function bounding techniques to obtain upper bounds on the bit error rate (BER) for maximum-likelihood (ML) decoding of turbo codes constructed with uniform interleavers. We apply the techniques to parallel concatenated coding (PCC) schemes with two-branch receive diversity. We present the average bounds on the BER for two cases: (a) fully interleaved channels with channel state information (CSI) and (b) Rayleigh fading channels with fading perfectly correlated in time. We show that as the fades in the two antennas become more correlated, the BER improvement obtained by increasing the block size N becomes larger. We also show that the effects of fading correlation for maximal ratio combining (MRC) and for equal gain combining (EGC) are almost identical, while those for selection combining (SC) are much more pronounced than for these two techniques.

I. Introduction

Recently, turbo codes have attracted much attention, because they have been shown to perform near the capacity limit on additive white Gaussian (AWGN) channels [1]-[4]. Turbo codes also perform well over wireless channels with either Rayleigh fading [5] or Rician fading [6]. In [5] the performance of turbo codes using coherent BPSK signaling has been investigated both on fully interleaved and exponentially correlated Rayleigh fading channels. In [6] turbo coding with a space diversity and maximal ratio combining (MRC) has been considered on Rician fading channels, and it has been shown that the space diversity is essential on slow Rician fading channels.

Among the many diversity schemes available, two-branch schemes are desirable, because they can achieve high diversity gain while maintaining compact size and reasonable complexity. Typically, as the number of antennas is increased beyond two, the incremental diversity gain decreases, so that most practical mobile receivers will employ only two branches. The prevalent space diversity combining schemes are selection combining (SC), maximal ratio combining (MRC) and equal gain combining (EGC). To gain sufficiently from the use of diversity, there must be a sufficient degree of statistical independence in the fading of the received signals in each of the diversity antennas. The assumption of statistical independence is valid only if the antennas are sufficiently separated. In mobile radio systems the signals at the mobile station become correlated as the antenna separation decreases. However, the effects of fading correlation on performance of turbo codes have not been clarified.

In this paper, we analyze the effects of fading correlation on the performance of turbo codes with two-branch receive diversity combining schemes. We use transfer function bounding techniques to obtain upper bounds on the bit error rate (BER) for maximum-likelihood (ML) decoding of turbo codes constructed with uniform interleavers. We apply the techniques to parallel concatenated coding (PCC) schemes with two-branch receive diversity. We present the average bounds on the BER for two cases: (a) fully interleaved channels with channel state information (CSI), that is, known fading amplitudes, and (b) Rayleigh fading channels with fading perfectly correlated in time. We show that as the fades in the two antennas become more correlated, the BER improvement obtained by increasing the block size N becomes larger. We also show that the effects of fading correlation for MRC and for EGC are almost identical, while those for SC are much more pronounced than for these two techniques.

II. System Model

We consider coherent BPSK signaling over Rayleigh slow-fading channels. With appropriate sampling, the discrete representation of this channel at the ith antenna is $y_{i,k} = a_{i,k}x_{i,k} + n_{i,k}$ where k is a time index, $x_{i,k}$ is a BPSK symbol amplitude ($\pm\sqrt{E_s}$), and $n_{i,k}$ is an i.i.d. AWGN component with zero mean and power spectral density $N_0/2$. The fading amplitude $a_{i,k}$ is modeled with a Rayleigh probability density function (pdf):

$$p(a_{i,k}) = \frac{a_{i,k}}{\sigma^2} e^{\frac{-a_{i,k}^2}{2\sigma^2}} \qquad \text{for } a_{i,k} > 0 \qquad (1)$$

where σ^2 represents the variance of in-phase and quadrature components $a_{i,k,I}$ and $a_{i,k,Q}$. These are Gaussian random variables and are related to $a_{i,k}$ by

$a_{i,k} = \sqrt{a_{i,k,I}^2 + a_{i,k,Q}^2}$. With sufficient channel interleaving (fully interleaved), the fading amplitudes on each antenna are independent in time: (i.e., $a_{i,k}$ and $a_{i,k'}$ are independent for $k \neq k'$.).

To gain sufficiently from the use of diversity, there must be a sufficient degree of statistical independence in the fading of the received signals in each of the diversity antennas. The assumption of statistical independence is valid only if the antennas are sufficiently separated. In mobile radio systems the signals at the mobile station become correlated as the antenna separation decreases. In the following analysis we analyze the effects of correlation between the fading amplitudes at two antennas on the performance of turbo codes with two-branch receive diversity combining schemes.

III. Theoretical Analysis

Here we use the transfer function bounding technique to obtain upper bounds on the BER for maximum-likelihood (ML) decoding of turbo codes [2]-[4]. As for convolutional codes, the error probability is upper-bounded by a union bound that sums contributions from error paths of different encoded weights, and the state diagram of the code is used to enumerate the paths of each possible weight. However, there are some differences between the transfer-function bounds for turbo codes and those for convolutional codes [2]. First, the bounds for turbo codes require a joint enumerator for all possible combinations of input weights and output weights of error events. Second, since turbo codes are block codes, it is difficult to enumerate compound error events that include more than one excursion from the all-zero state within the fixed block length. Third, since it is difficult to obtain the bound for any particular choice of permutation, the bound is developed as an average over randomly chosen interleavers. Finally, the bounds are derived as upper bounds on the BER of a ML decoder. These bounds are known to be useful for evaluating the error performance where the computer simulations are not feasible. Here, we apply this bound to the correlated Rayleigh fading channel.

We consider the union upper bound for the ML decoding of an (N, K) block code. Without loss of generality, the uniform error property applies, and the probability of word error can be written as

$$P_W \leq \sum_{d=1}^{N} A(d) P_2(d) \tag{2}$$

where $A(d)$ is the number of codewords with Hamming weight d and $P_2(d)$ is the error event probability. To alleviate the difficulty of derivation of $A(d)$, some researcher have developed an average upper bound constructed by averaging over all possible interleavers [2]-[4]. The average weight distribution can be written as

$$\bar{A(d)} = \sum_{i=1}^{K} \binom{K}{i} p(d|i) \tag{3}$$

where $\binom{K}{i}$ is the number of input words with Hamming weight i and $p(d|i)$ is the probability that an input word with Hamming weight i produces a codeword with Hamming weight d. The average upper bound on the probability of word and bit error can be written as

$$P_b \leq \sum_{i=1}^{K} \binom{K}{i} E_{d|i} \left[P_2(d) \right], \tag{4}$$

$$P_b \leq \sum_{i=1}^{K} \frac{i}{K} \binom{K}{i} E_{d|i} \left[P_2(d) \right]. \tag{5}$$

Here, $E_{d|i}[\cdot]$ is an expectation with respect to $p(d|i)$.

We analyze the system with two-branch receive diversity in a correlated Rayleigh fading channel. At the receiver ideal CSI is assumed. SC, MRC, and EGC are considered as diversity combining techniques.

A. Fully Interleaved Channels With CSI

On the fully interleaved channel with CSI, the exact probability of incorrectly decoding a codeword $\boldsymbol{c}_{\mathrm{o}}$ into a codeword $\boldsymbol{c}_d$ that differs from $\boldsymbol{c}_{\mathrm{o}}$ in d bit positions can be written as

$$P(\boldsymbol{c}_{\mathrm{o}} \rightarrow \boldsymbol{c}_d | \boldsymbol{b}) = Q\left(\sqrt{\frac{2E_s}{N_0} \sum_{k=1}^{d} b_k^2} \right) \tag{6}$$

where b_k is a channel gain after diversity combining and $Q(\cdot)$ is the tail integral of a standard Gaussian density with zero mean and unit variance defined as

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz. \tag{7}$$

Averaging $P(\boldsymbol{c}_{\mathrm{o}} \rightarrow \boldsymbol{c}_d | \boldsymbol{b})$ over the channel gains $\boldsymbol{b}$, the error event probability can be written as

$$P_2(d) = \int_{b_1} \cdots \int_{b_d} p(\boldsymbol{b}) P(\boldsymbol{c}_{\mathrm{o}} \rightarrow \boldsymbol{c}_d | \boldsymbol{b}) db_1 \cdots db_d \tag{8}$$

and

$$p(\boldsymbol{b}) = \prod_{i=1}^{d} p(b_i). \tag{9}$$

It has been shown that the spacing between antenna elements can influence the cross-correlation between the fading signal envelopes (ξ^2), and that the effective signal power is scaled by $(1-\xi^2)^{1/2}$ [7]. This paper assumes the signals received on two antennas to be subject to mutually correlated Rayleigh fading with a symmetrical power spectrum. The fading power spectra of the two received signals are assumed to be identical.

To alleviate the difficulty of the exact evaluation of eq. (8), we use the following upper bound on the Q-function in the following analysis:

$$Q(\sqrt{x}) \leq \frac{1}{2}e^{-x/2}, \qquad x \geq 0. \tag{10}$$

SC selects the branch having the largest signal envelope. The value of b_k is represented as $b_k = \max(a_{1,k}, a_{2,k})$. For small value of b_k, the pdf of b_k with SC in a correlated Rayleigh fading channel can be approximated as [8]

$$p(b_k) \approx \frac{b_k}{\sigma^2}\frac{2}{1+\xi^2}\left(e^{-\frac{b_k^2}{2\sigma^2}} - e^{-\frac{2}{1-\xi^2}\frac{b_k^2}{2\sigma^2}}\right) \tag{11}$$

where σ^2 is the average signal power of a single branch. Substituting eq. (11) into eq. (8), the product of integrals, for two-branch receive diversity with SC in a correlated Rayleigh fading channel with CSI, simplifies the bound to

$$P_2(d) \leq \frac{1}{2}\left[\frac{1}{(1+\xi^2)\sigma^2}\left(\frac{1}{\frac{1}{2\sigma^2}+\frac{E_s}{N_0}} - \frac{1}{\frac{2}{1-\xi^2}\cdot\frac{1}{2\sigma^2}+\frac{E_s}{N_0}}\right)\right]^d. \tag{12}$$

MRC cophases, amplifies each signal by a factor proportional to its envelope and sums them. The value of b_k is represented as $b_k = \sqrt{a_{1,k}^2 + a_{2,k}^2}$. The pdf of b_k with MRC in a correlated Rayleigh fading channel is given by [7]

$$p(b_k) = \frac{b_k}{2\xi^2\sigma^2}\left(e^{-\frac{b_k^2}{2\sigma^2(1+\xi^2)}} - e^{-\frac{b_k^2}{2\sigma^2(1-\xi^2)}}\right). \tag{13}$$

Substituting eq. (13) into eq. (8), the product of integrals, for two-branch receive diversity with MRC in a correlated Rayleigh fading channel with CSI, simplifies the bound to

$$P_2(d) \leq \frac{1}{2}\left[\frac{1}{4\xi^2\sigma^2}\left(\frac{1}{\frac{1}{2\sigma^2(1+\xi^2)}+\frac{E_s}{N_0}} - \frac{1}{\frac{1}{2\sigma^2(1-\xi^2)}+\frac{E_s}{N_0}}\right)\right]^d. \tag{14}$$

A good approximation to the pdf of EGC can be obtained using that of MRC, and is obtained by replacing the average signal power σ^2 of a single branch with $(\sqrt{3}/2)\sigma^2$ [7]. Thus, a good approximation to the error event probability of EGC can be obtained using that of MRC by replacing σ^2 with $(\sqrt{3}/2)\sigma^2$.

B. Rayleigh Fading Channels With Fading Perfectly Correlated in Time

Next we consider the channel to be fully correlated in time and the fading amplitude is constant over the entire block length N. In this case eq. (8) becomes

$$P_2(d) = \int_0^\infty p(b)Q\left(\sqrt{\frac{2db^2E_s}{N_0}}\right)db \tag{15}$$

where the channel gain after diversity combining over the entire block length N is assumed to be constant b. The product of integrals, for two-branch receive diversity with SC in a fully correlated Rayleigh fading channel in time with CSI, simplifies the bound to

$$P_2(d) \leq \frac{1}{2\sigma^2(1+\xi^2)}\left(\frac{1}{\frac{1}{2\sigma^2}+d\frac{E_s}{N_0}} - \frac{1}{\frac{2}{1-\xi^2}\cdot\frac{1}{2\sigma^2}+d\frac{E_s}{N_0}}\right). \tag{16}$$

The product of integrals, for two-branch receive diversity with MRC in a fully correlated Rayleigh fading channel in time with CSI, simplifies the bound to

$$P_2(d) \leq \frac{1}{8\sigma^2\xi^2}\left(\frac{1}{\frac{1}{2\sigma^2(1+\xi^2)}+d\frac{E_s}{N_0}} - \frac{1}{\frac{1}{2\sigma^2(1-\xi^2)}+d\frac{E_s}{N_0}}\right). \tag{17}$$

A good approximation to EGC can be also obtained using that of MRC by replacing the average signal power σ^2 of a single branch with $(\sqrt{3}/2)\sigma^2$ [7]. Thus, a good approximation to the error event probability of EGC can be also obtained using that of MRC by replacing σ^2 with $(\sqrt{3}/2)\sigma^2$.

IV. Numerical Results

We present some bounds on BER for the (1,5/7,5/7) code with the various values of the cross-correlation between the fading signal envelopes ξ^2 for BPSK signal sets. We assume a normalized Rayleigh fading channel on each antenna, that is, σ^2 is set to 1/2.

Figs. 1-3 show the bounds on P_b for the (1,5/7,5/7) code with some values of the block length N and the fading cross-correlation ξ^2 for two-branch receive diversity schemes with SC, MRC, and EGC, respectively. These results assume a fully interleaved correlated Rayleigh fading channel with CSI, respectively.

Table 1. E_b/N_0 required to achieve $P_b \leq 10^{-5}$ for the (1,5/7,5/7) code with some values of the block length N for BPSK with two-branch SC, MRC and EGC diversities. These results assume a fully interleaved correlated Rayleigh fading channel with CSI.

ξ^2	SC [dB]		MRC [dB]		EGC [dB]	
	N=10	N=100	N=10	N=100	N=10	N=100
0.00	6.0	3.6	4.6	2.2	5.3	2.9
0.50	7.1	4.6	4.9	2.5	5.6	3.1
0.95	9.0	6.1	6.1	3.2	6.7	3.8
1.00	9.4	6.3	6.3	3.3	7.0	3.9

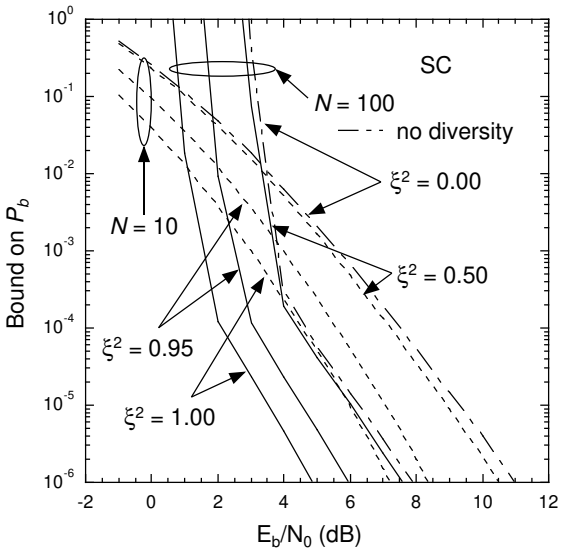

Fig. 1: Bounds on P_b for the (1,5/7,5/7) code with various fading cross-correlation ξ^2 for two-branch receive diversity schemes with SC on fully interleaved channels with CSI.

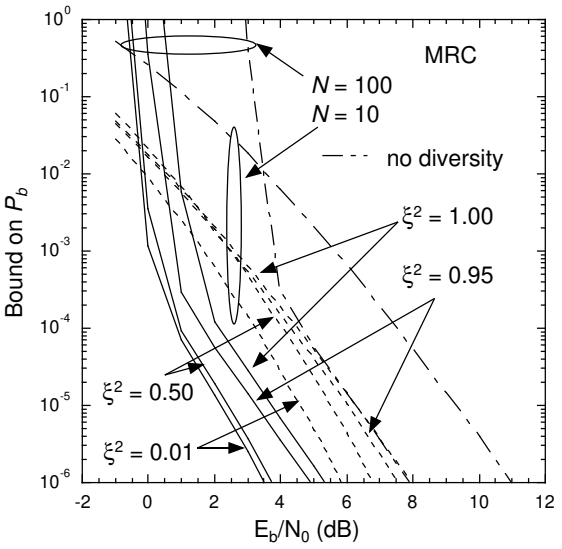

Fig. 2: Bounds on P_b for the (1,5/7,5/7) code with various fading cross-correlation ξ^2 for two-branch receive diversity schemes with MRC on fully interleaved channels with CSI.

As same as for no diversity scheme on AWGN channels reported in [2]-[4], the bounds on P_b for each diversity combining scheme also abruptly become useless roughly below a threshold value of the information bit energy-to-noise ratio E_b/N_0 determined by the computational cutoff rate R_0 if the block length N is large. Note that in Fig. 1, the required E_b/N_0 for two-branch receive diversity with SC when ξ^2 =1.0 is equal to that of the system without diversity. In all cases, the required E_b/N_0 increases as ξ^2 increases, because the increase of ξ^2 results in the reduction of the effective signal power with two branch receive diversity. We see that the effects of fading correlation for MRC and EGC are almost identical, while those for SC are much more pronounced. For example, increasing ξ^2 from 0.00 to 0.50 degrades MRC and EGC by about 0.3 dB, while it degrades SC by about 1.0 dB. Thus, while the simplicity of SC makes this technique desirable, one should be cautious about applying the technique when fading correlation is significant.

Table 1 shows the E_b/N_0 required to achieve $P_b \leq 10^{-5}$ for the (1,5/7,5/7) code with some values of the block length N for BPSK with two-branch SC, MRC, and EGC diversities. These results assume a fully interleaved correlated Rayleigh fading channel with CSI. We can confirm in all cases that the required E_b/N_0 increases as ξ^2 increases, because the increase of ξ^2 results in the reduction of the effective signal power with two branch receive diversity. However, as ξ^2 increases, the BER improvement obtained by increasing N becomes larger. Thus, turbo codes are effective in practical wireless communications. We can also confirm again that the effects of fading correlation for MRC and for EGC are almost identical, while those for SC are much more pronounced than for these two techniques.

Fig. 4 shows the bounds on P_b for the (1,5/7,5/7) code with some values of the fading cross-correlation ξ^2 for two-branch receive diversity schemes with SC, MRC, and EGC, respectively. These results assume Rayleigh fading channels with fading perfectly correlated in time where the block length N is 10. Note that the performance on Rayleigh fading channels with fading perfectly correlated in time results in the upper bound on that on fully interleaved channels

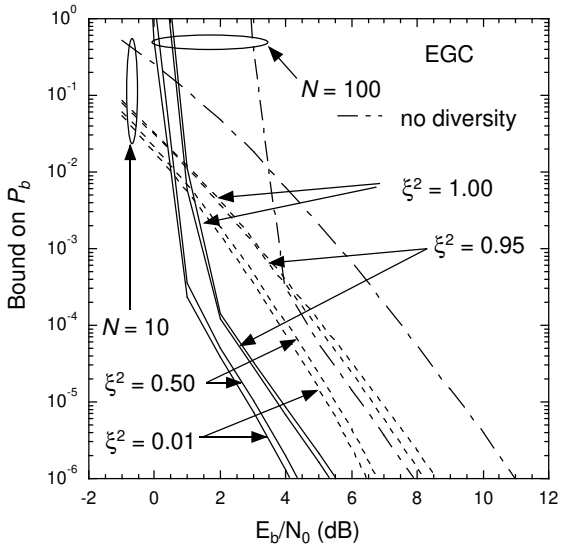

Fig. 3: Bounds on P_b for the (1,5/7,5/7) code with various fading cross-correlation ξ^2 for two-branch receive diversity schemes with EGC on fully interleaved channels with CSI.

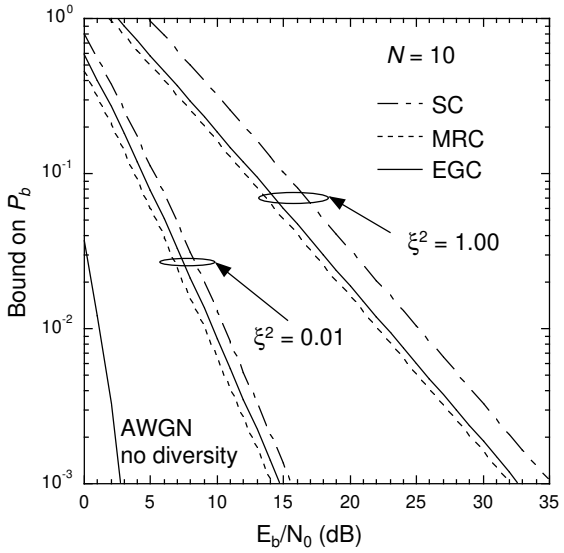

Fig. 4: Bounds on P_b for the (1,5/7,5/7) code with some fading cross-correlation ξ^2 for two-branch receive diversity schemes with SC, MRC, and EGC on Rayleigh fading channels with fading perfectly correlated in time where N=10.

with CSI. Note again that the required E_b/N_0 for two-branch receive diversity with SC when ξ^2 =1.0 is equal to that of the system without diversity. In Fig. 4 we can see the trends similar to those seen on the fully interleaved channels with CSI; the required E_b/N_0 increases as ξ^2 increases and the effects of fading correlation for SC is larger than those for MRC and EGC and the effects of fading correlation for MRC and for EGC are almost identical, while those for SC are much more pronounced than for these two techniques.

V. Conclusions

We analyzed the effects of fading correlation on the performance of turbo codes with two-branch receive diversity combining schemes. We have obtained the upper bounds on the BER for ML decoding of turbo codes constructed with uniform interleavers. We presented the average bounds on the BER for two cases: (a) fully interleaved channels with channel state information (CSI) and (b) Rayleigh fading channels with fading perfectly correlated in time. We showed that as the fades in the two antennas become more correlated, the BER improvement obtained by increasing the block size N becomes larger. We also showed that the effects of fading correlation for MRC and for EGC are almost identical, while those for SC are much more pronounced than for these two techniques.

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