Wavelets

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Wavelets: Who Gives A Damn?

- The Fourier Transform
  \[ \hat{f}(s) = \int x(t) e^{-2\pi i s t} dt \]
  - Has no time resolution (integral is from \(-\infty\) to \(\infty\))
  - Not very useful on non-stationary signals
Solution #1: Short Term FT

- Short Term Fourier Transform
  
  \[
  \text{STFT}(t', f) = \int x(t) \cdot e^{-2\pi i tf} \cdot w(t-t') dt
  \]

- Computes FT over a small window
- Window is usually a Gaussian
- Cannot provide localization in both time and frequency
Visual Example of STFT
The Uncertainty Principle

- $4\pi \Omega \cdot T \geq 1$

- As frequency resolution increases, time resolution decreases (and vice versa)
Solution #2: Wavelets

1. A set of building blocks (set of basis functions)
   - All functions generated from the “mother wavelet” by scaling and translation

2. Satisfy the multiresolution condition
   - By making expansion signals half as wide, we can represent a larger set of functions (including the original set)

3. Provide time-frequency localization

4. Coefficients can be calculated efficiently
   - For many wavelet systems, requires $O(N)$ time (FFT requires $O(N \cdot \log(N))$)
Commonly Used Wavelets

- **Haar Wavelet**
  \[ \psi(x) = \begin{cases} 
  1, & 0 < x < \frac{1}{2} \\
  -1, & \frac{1}{2} < x < 1 \\
  0, & \text{otherwise} 
\end{cases} \]

- **Morlet Wavelet**
  \[ \psi(x) = \cos(5x) \cdot e^{-\frac{x^2}{2}} \]

- **Mexican Hat Wavelet**
  \[ \psi(x) = (1 - x^2) \cdot e^{-\frac{x^2}{2}} \]
Representing A Function As A Sum

- Fourier series use sine and cosine waves
- Wavelets can be used instead
  - \( f(t) = \sum_k \sum_j [a_{j,k} \cdot \psi_{j,k}(t)] \)
  - where: \( \psi_{j,k}(t) = 2^{j/2} \psi(2^j t - k) \)
- Problems with Fourier series that may be avoided by wavelets:
  - Non-stationary signals
  - Signals with discontinuities
Approximating a function using Haar Wavelets
The Continuous Wavelet Transform

\[ CWT(\tau, s) = \int_{-\infty}^{\infty} x(t) \cdot s^{-\frac{1}{2}} \psi^* \left( \frac{t-\tau}{s} \right) dt \]

- Allows us to find the amplitude of “frequency” components at different times
- Plotted in 3 dimensions (amplitude, scale, translation)

Differences with Short Term FT

- Different window sizes used to measure different frequencies
Example CWT

A non-stationary signal and its corresponding Continuous Wavelet Transform
Applications

- Data Compression
  - FBI fingerprint archive (200+ terabytes)
- Denoising Noisy Data
- Storing/Synthesizing Musical Tones
Data Compression / Denoising

- Both tasks involve removing insignificant information
- Procedure
  1. Choose a wavelet type and decomposition level. Compute decomposition of signal.
  2. Remove coefficients less than a given threshold
  3. Reconstruct signal (for denoising only)
Wavelet Image Compression

Wavelets | JPEG

Compression Ratio – 5:1
Wavelet Image Compression

Wavelets

JPEG

Compression Ratio – 30:1
Wavelet Image Compression

Wavelets

JPEG

Compression Ratio – 50:1
Wavelet Image Compression

Wavelets

Compression Ratio – 80:1
## Recap

### Wavelets
- Multiple wavelet types
- Wavelet Expansion
- Continuous Wavelet Transform
- Good time/frequency localization

### Fourier
- Complex exponentials
- Fourier Series
- Fourier Transform
- Poor time/frequency localization
Questions?