On the Statistics of Intrachannel Four-Wave Mixing in Phase-Modulated Optical Communication Systems

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***Abstract*—We have analytically derived the correlation functions of intrachannel four-wave mixing (IFWM)-induced phase and amplitude noises in phase-modulated optical communication systems. The phase and amplitude noises are correlated with each other for binary phase-shift keying (PSK) systems but uncorrelated for M-ary PSK systems with $M > 2$. We have also derived analytical approximations to the probability distribution of IFWM-induced phase noise for PSK and differential PSK systems. Furthermore, we have studied the performance of an optimal linear phase-noise predictor derived from the IFWM-induced phase-noise autocorrelation function. This yields a performance improvement of 1.8 dB when IFWM-induced phase noise is the dominant impairment, and an improvement of 0.8 to 1.2 dB in the presence of amplified spontaneous emission noise and nonlinear phase noise in typical terrestrial links.**

***Index Terms*—Intersymbol interference, optical fiber communications, optical Kerr effect, phase modulation, phase noise.**

I. Introduction

OPTICAL fiber transmission systems using coherent or differentially coherent detection of phase-modulated signals, such as phase-shift keying (PSK), differential PSK (DPSK), or quadrature-amplitude modulation (QAM), are subject to impairment by phase noise. Laser phase noise and amplified spontaneous emission (ASE) from inline amplifiers are major source of phase noise. In presence of Kerr nonlinearity, two other mechanisms contribute to additional phase noise: 1) the Gordon–Mollenauer Effect, resulting from the interaction of ASE noise with Kerr nonlinearity, and 2) intrachannel four-wave mixing (IFWM), a form of nonlinear intersymbol interference (ISI) resulting from the interaction of Kerr nonlinearity and chromatic dispersion. The relative impact of nonlinear phase noise and IFWM-induced phase noise depends on the symbol rate and on the dispersion map used in the system [1], [2]. At present, IFWM is generally regarded as noise because its structure is much more complicated than linear ISI.

Signal perturbations by IFWM were first observed by Shake [3] and Essiambre *et al.* [4] for on–off keying (OOK) systems in the form of amplitude fluctuations in the "1" bit slots and ghost pulse generation in the "0" bit slots. Mecozzi presented the first analytical study of IFWM using first-order perturbation theory [5]. IFWM can be mitigated by symmetric dispersion maps [6]–[8] or appropriate dispersion precompensation [9], alternating polarization [10], [11] phase modulation [12]–[15], coding [16]–[18], and optical phase conjugation [19]. However, most of the studies in the literature focus on IFWM-induced amplitude noise in OOK systems. IFWM-induced phase noise for DPSK systems was first studied by Wei and Liu [20], who found from simulations that in DPSK systems, the IFWM-induced phase noise is correlated from symbol to symbol. In addition, Ho [21] presented semianalytical methods to characterize bit error ratio (BER) performance based on empirical probability distributions of IFWM. However, no analytical study has been undertaken to better understand the statistical properties of IFWM for phase-modulated systems. In addition, it has been known for some time that in PSK or DPSK systems at symbol rates above 10 GSym/s, intrachannel nonlinearities dominate over interchannel effects such as cross-phase modulation (XPM) [22] and therefore IFWM is a more fundamental problem that needs to be addressed.

In this paper, we derive the correlation properties of IFWM-induced phase and amplitude noises and show that they are correlated with each other for BPSK systems but uncorrelated for M-ary PSK systems with $M > 2$. Furthermore, we provide analytical approximations to the probability distribution of IFWM-induced phase noise for PSK and DPSK systems. The theories developed are in close agreement with simulation results. We then exploit the analytical knowledge of the IFWM-induced phase-noise correlation by studying the performance gain using an optimal linear phase-noise prediction filter. This yields a performance improvement of 1.8 dB when IFWM-induced phase noise is the dominant impairment and an improvement of 0.8 to 1.2 dB in the presence of ASE noise and nonlinear phase noise.

The rest of this paper is organized as follows. In Section II, we derive the correlation of IFWM-induced phase and amplitude noises and show that the correlation structure is different for binary and M-ary systems with $M > 2$. In Section III, we present analytical approximations to the probability distribution of IFWM-induced phase noise for PSK and DPSK systems. In Section IV, we derive an optimal linear phase-noise predictor based on the IFWM-induced phase-noise correlation and study its performance in typical terrestrial links.

II. Correlation Properties of IFWM

Consider a single-channel system with phase-modulated pulse trains transmitted through a multispan link with perfect

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dispersion compensation per span. The signal at the transmitter is given by

$$E(0,t) = \sum_k x_k u(0, t-kT) = \sum_k x_k u_k \tag{1}$$

where $\mathbf{x}_k = \{1, e^{j2\pi/M} \ldots e^{j2\pi(M-1)/M}\}$ are M-ary phase-modulated signals, $u(z,t)$ is the pulse shape, α is the attenuation coefficient, and T is the symbol period. Signal propagation in optical fiber is described by the nonlinear Schrödinger equation [25]

$$\frac{\partial E}{\partial z} - \frac{j\beta_2(z)}{2}\frac{\partial^2 E}{\partial t^2} + \frac{\alpha(z)}{2}E = j\gamma(z)|E|^2 E \tag{2}$$

where β_2 is the chromatic dispersion parameter and γ is the fiber nonlinear coefficient. As in other related works in the literature, we neglect the effect of polarization mode dispersion in this analysis. Using first-order perturbation theory to approximate the solution of the nonlinear Schrödinger equation, Mecozzi [5] showed that one can decompose the transmitted signal into

$$E \approx E^{(l)} + \Delta E = \sum_k x_k u_k^{(l)} + \Delta u_k$$

where $E^{(l)}$ denotes the linear solution to (2) (obtained by setting $\gamma = 0$) and ΔE denotes the nonlinear perturbation. The perturbation term satisfies

$$\begin{aligned}\frac{\partial \Delta E}{\partial z} &- \frac{j\beta_2(z)}{2}\frac{\partial^2 \Delta E}{\partial t^2} + \frac{\alpha(z)}{2}\Delta E \\ &= j\gamma(z)\left|E^{(l)}\right|^2 E^{(l)} \\ &= j\gamma(z)\sum_{l,m,p} x_l x_m x_p^* u_l^{(l)} u_m^{(l)} u_p^{(l)*}.\end{aligned} \tag{3}$$

With high local dispersion, it can be shown that the terms from the right-hand side of (3) that contribute to Δu_k are those with

$$l + m - p = k \tag{4}$$

and therefore these nonlinear signal perturbations are termed IFWM, as (4) is analogous to the relation of the signal frequencies involved in a four-wave mixing process. We emphasize that IFWM, and other so-called intrachannel nonlinearities, originate from self-phase modulation (SPM), not FWM. Focusing on the pulse at bit slot 0 without loss of generality, the nonlinear perturbations to $u_0^{(l)}$ originate from

$$j\gamma \sum_{l,m} x_l x_m x_{l+m}^* u_l^{(l)} u_m^{(l)} u_{l+m}^{(l)*}$$

from (3). If we assume $C'_{l,m}(t)$ as the solution to

$$\frac{\partial C'_{l,m}}{\partial z} - \frac{j\beta_2(z)}{2}\frac{\partial^2 \Delta C'_{l,m}}{\partial t^2} + \frac{\alpha(z)}{2}C'_{l,m} = j\gamma(z) u_l^{(l)} u_m^{(l)} u_{l+m}^{(l)*} \tag{5}$$

by linearity, the overall nonlinear perturbations is then $\Delta u_0 = \sum_{l,m} C'_{l,m}(t)$. As the x_is are constant-intensity signals, terms within Δu_0 with either l or $m = 0$ are deterministic perturbations and in the literature are called intrachannel SPM (ISPM) (when both $l = m = 0$) and intrachannel cross-phase modulation (IXPM) (when either l or $m = 0$). If we denote $u_0^{(S+X)}$ as the collective deterministic perturbations by ISPM and IXPM to $u_0^{(l)}$, the IFWM-induced phase noise can be approximated as

$$\begin{aligned}\phi_0(t) &\approx \sum_{l,m\neq 0} \operatorname{Im}\left\{\frac{x_l x_m x_{l+m}^* C'_{l,m}(t)}{x_0\left(u_0^{(l)} + u_0^{(S+X)}\right)}\right\} \\ &= \sum_{l,m\neq 0} \operatorname{Im}\left\{x_l x_m x_{l+m}^* x_o^* C_{l,m}\right\}\end{aligned} \tag{6}$$

and the IFWM-induced amplitude noise normalized with respect to the signal amplitude is

$$\Delta r_0(t) \approx \sum_{l,m\neq 0} \operatorname{Re}\left\{x_l x_m x_{l+m}^* x_o^* C_{l,m}\right\}. \tag{7}$$

The approximations are valid when the noises are small compared to the signals. Note that $C_{l,m}$ is a function of t, α, β_2, and γ but is abbreviated here for simplicity. It can be easily shown that the terms in $\phi_0(t)$ (and similarly in $\Delta r_0(t)$) are uncorrelated. For the case of binary PSK (BPSK) signals, the variances of ϕ and Δr are given by

$$\begin{aligned}\sigma_\phi^2(t) &= \frac{1}{2}\sum_{l,m}\left(\operatorname{Im}\{C_{l,m}\}\right)^2 \\ \sigma_{\Delta r}^2(t) &= \frac{1}{2}\sum_{l,m}\left(\operatorname{Re}\{C_{l,m}\}\right)^2.\end{aligned}$$

For an M-PSK system with $M > 2$, it can be shown that

$$\sigma_{\Delta r}^2(t) = \sigma_\phi^2(t) = \frac{1}{2}\sum_{l,m}|C_{l,m}|^2.$$

Now, define

$$A = x_l x_m x_{l+m}^* x_0^* \quad \text{and} \quad B = x_p x_q x_{p+q-k}^* x_k^*.$$

The autocorrelation function of the IFWM-induced phase noise $R_\phi(k,t) = E[\phi_0(t)\phi_k(t+kT)]$ is given by

$$\begin{aligned}R_\phi(k,t) &= E\left[\phi_0(t)\phi_k(t+kT)\right] \\ &= -\frac{1}{4}E\left[\sum_{l,m}\left(x_l x_m x_{l+m}^* x_0^* C_{l,m} - c.c.\right)\right. \\ &\qquad \left.\times \sum_{p,q}\left(x_p x_q x_{p+q-k}^* x_k^* C_{p-k,q-k} - c.c.\right)\right] \\ &= -\frac{1}{4}\sum_{l,m}\sum_{p,q}\left\{E[C_{l,m}C_{p-k,q-k}AB + c.c.]\right. \\ &\qquad \left. - E\left[C_{l,m}C_{p-k,q-k}^* AB^* + c.c.\right]\right\}\end{aligned} \tag{8}$$

where $c.c.$ denotes complex conjugate. Since the x_is in an M-ary system are zero-mean independent and identically distributed (i.i.d.) random variables with $E[|x_i|^{d_1}] = E[x_i^M] = E[(x_i^*)^M] = 1$ for any $d_1 \in \mathbb{Z}$

$$E\left[x_i^{d_2} x_j^{d_3}\right] = E\left[x_i^{d_2}\right]E\left[x_j^{d_3}\right] = \begin{cases}1, & d_2 = d_3 = M \\ 0, & \text{otherwise}\end{cases}. \tag{9}$$

With this insight, one can significantly simplify (8). First, the expectations of the terms in (8) and their complex conjugates are equal. The cases where the expectations do not vanish occur when the indexes l, m, p, q are such that AB or AB^* consist of factors of $(\cdot)^M$ and/or $|\cdot|^{d_1}$. Now

$$\sum_{l,m\neq 0}\sum_{p,q\neq k} E[AB] = \sum_{m\neq 0} E\left[|x_0|^2|x_k|^2|x_m|^2|x_{k+m}|^2\right] \quad (10)$$

when $p=0, l=k$, and $q=m+k$. Notice that the quadruple summation reduces to a single summation. Furthermore

$$\sum_{l,m\neq 0}\sum_{p,q\neq k} E[AB^*] = \sum_{m\neq 0,k} E\left[|x_0|^2|x_k|^2|x_m|^2|x_{k-m}|^2\right] \quad (11)$$

when $l+m=k$, $p+q=k$, and $l=p$. For quadrature PSK (QPSK) systems, it can be shown that there exist no combinations of l, m, p, and q such that AB, $AB^* = (\cdot)^4(\cdot)^4$ or $(\cdot)^4|\cdot|^2|\cdot|^2$ and $R_\phi(k,t)$ only consists of terms described in (10) and (11). Therefore

$$R_\phi(k,t) = \frac{1}{2}\sum_m \mathrm{Re}\left\{C_{m,k-m}C^*_{-m,m-k}\right\} - \frac{1}{2}\sum_m \mathrm{Re}\{C_{m,k}C_{m,-k}\}. \quad (12)$$

The derivation of $R_\phi(k,t)$ for BPSK systems is more complicated, as there exist various combinations of l, m, p, and q such that $ABAB^* = (\cdot)^2(\cdot)^2(\cdot)^2(\cdot)^2 = 1$. In particular

$$\sum_{l,m\neq 0}\sum_{p,q\neq k} E[AB] = \sum_{m\neq 0,k} E\left[(x_0)^2(x_k)^2(x_m)^2(x_{k-m})^2\right] \quad (13)$$

when $l+m=k$, $p+q=k$, $l=p$, and

$$\sum_{l,m\neq 0}\sum_{p,q\neq k} E[AB^*] = \sum_{m\neq 0} E\left[|x_0|^2|x_k|^2|x_m|^2|x_{k+m}|^2\right] \quad (14)$$

when $p=0, l=k$, and $q=m+k$. The autocorrelation function $R_\phi(k,t)$ for BPSK systems is then given by

$$\begin{aligned} R_\phi(k,t) = &\frac{1}{2}\sum_m \mathrm{Re}\left\{C_{m,k-m}C^*_{-m,m-k}\right\} \\ &- \frac{1}{2}\sum_m \mathrm{Re}\{C_{m,k}C_{m,-k}\} \\ &+ \frac{1}{2}\sum_m \mathrm{Re}\left\{C_{m,k}C^*_{m,-k}\right\} \\ &- \frac{1}{2}\sum_m \mathrm{Re}\{C_{m,k-m}C_{-m,m-k}\}. \quad (15)\end{aligned}$$

Fig. 1 compares the autocorrelation function given by (12) and that obtained from simulations of a 32-pulse sequence propagating in a single span of a typical terrestrial link. The system parameters are listed in Table I and will be used for the rest of this paper. The symbol rate is 40 GSym/s with $T = 25$ ps. The pulses are 33% return-to-zero (RZ)-QPSK with Gaussian pulse shape and the average launched signal power is 0 dBm. In Fig. 1, a noiseless lumped amplifier is assumed at the end of the span. It should be noted that $R_\phi(k,t)$ simply scales with the square of the number of spans in a multispan system. As it is not possible to exhaust all the possible $4^{32} \sim 2\times 10^{19}$ bit sequences, 5000 random trials of the 32-bit sequence are used in the simulation to obtain the empirical estimate of the autocorrelation function. From the figure, it can be shown that the analytical predictions of (12) agree well with simulation results. In addition, for each k, $R_\phi(k,t)$ increases with t, as the IFWM pulse is $\sqrt{3}$ times wider than the signal pulse, which leads to larger phase fluctuation for large t [26].

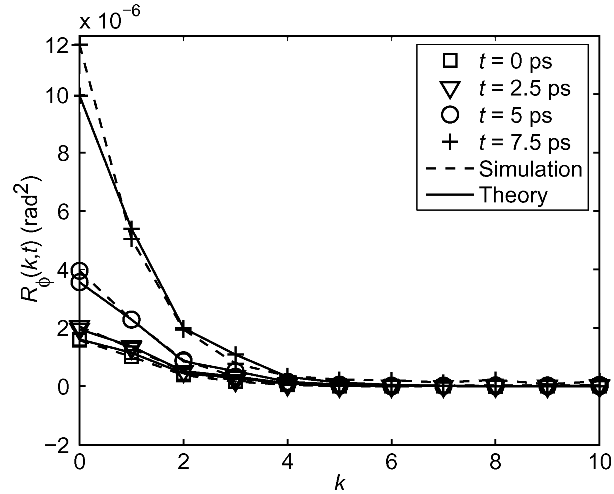

Fig. 1. Autocorrelation of IFWM-induced phase noise for a 40 GSym/s RZ-QPSK system after propagation through one span of fiber. Average launched power is 0 dBm.

TABLE I
FIBER PARAMETER VALUES USED IN THE SYSTEM

L_{SMF}	80 km	L_{DCF}	16 km
α_{SMF}	0.25 dB/km	α_{DCF}	0.6 dB/km
D_{SMF}	17 ps/nm-km	D_{DCF}	-85 ps/nm-km
γ_{SMF}	1.2 W^{-1}/km	γ_{DCF}	5.3 W^{-1}/km

The autocorrelation function of the normalized IFWM-induced amplitude fluctuations can be derived in a similar way and will be omitted here. For BPSK systems

$$\begin{aligned} R_{\Delta r_0}(k,t) = &\frac{1}{2}\sum_m \mathrm{Re}\left\{C_{m,k-m}C^*_{-m,m-k}\right\} \\ &+ \frac{1}{2}\sum_m \mathrm{Re}\{C_{m,k}C_{m,-k}\} \\ &+ \frac{1}{2}\sum_m \mathrm{Re}\left\{C_{m,k}C^*_{m,-k}\right\} \\ &+ \frac{1}{2}\sum_m \mathrm{Re}\{C_{m,k-m}C_{-m,m-k}\}. \quad (16)\end{aligned}$$

Similar to (15) and (12), the last two terms of (16) vanish for M-ary PSK systems with $M > 2$. Note the change in sign of two particular terms in (16) and (15). Numerical calculations show that those terms are the dominant ones in $R_\phi(k,t)$ and $R_{\Delta r}(k,t)$ and hence, for a given k, if the IFWM-induced phase noise has positive autocorrelation, the IFWM-induced amplitude noise generally has negative autocorrelation. Such conclusions are in agreement with the observations reported in the literature [21].

Finally, the correlation between the IFWM-induced phase and amplitude noises is given by

$$\begin{aligned} E\left[\phi_0(t)\Delta r_0(t)\right] &= E\left[\sum_{l,m\neq 0}\mathrm{Im}\{AC_{l,m}\}\sum_{l,m\neq 0}\mathrm{Re}\{AC_{l,m}\}\right] \\ &= \sum_{l,m\neq 0} E\left[\mathrm{Im}\{AC_{l,m}\}\mathrm{Re}\{AC_{l,m}\}\right] \\ &= \frac{1}{4j}E\left[A^2C_{l,m}^2 - \left(A^2C_{l,m}^2\right)^*\right] \\ &= \begin{cases} \frac{1}{2}\mathrm{Im}\left\{C_{l,m}^2\right\}, & \text{BPSK} \\ 0, & \text{MPSK} \end{cases}. \end{aligned} \tag{17}$$

Furthermore, it can be shown that $E[\phi_0(t)\Delta r_k(t+kT)] = 0$ for M-ary PSK systems. Therefore, the IFWM-induced phase and amplitude noises are uncorrelated, except for BPSK systems. Consequently, compensating ϕ_0 by adding a term proportional to the signal amplitude or power will not reduce the overall phase-noise variance. This is in contrast to the performance gained by such techniques in nonlinear phase-noise mitigation [27]. Nonetheless, ϕ_0 and Δr_0 are not independent of each other.

Fig. 2 shows scatter plots of ϕ_0 and Δr_0 sampled at $t = 0$ ps obtained from simulations for various phase-modulated systems. The phase and amplitude noises for a BPSK system are clearly negatively correlated. With QPSK modulation format, although ϕ_0 and Δr_0 are shown to be uncorrelated, their dependence is clearly illustrated by the asymmetry of the scatter plot. Furthermore, the distribution of Δr_0 is skewed towards positive values. As detection errors occur when $\Delta r_0 < -1$, such a skewness is beneficial for phase-modulated systems.

III. Probability Distribution of IFWM Phase Noise

Knowledge of the probability distribution of noise sources is important for characterizing the BER and other system performance metrics. Wei and Liu showed [28] that for a BPSK system with lossless fiber and nonsymmetric dispersion maps, IFWM-induced amplitude noise limits system performance, while IFWM-induced phase noise has a negligible effect on the BER. In realistic system designs, particularly when using QPSK, IFWM-induced amplitude noise is expected to be less important than several forms of phase noise, including IFWM-induced phase noise, nonlinear phase noise, and laser phase noise. There is currently no analytical knowledge of the probability distribution of ϕ_0 in the literature, and the BER performance of PSK or DPSK systems with IFWM has been obtained only by making use of empirical distributions of IFWM [21], [29]. In this section, we present an approximation to the probability distribution of IFWM-induced phase noise for M-ary PSK and DPSK systems.

It has already been shown that the terms in $\phi_0(t)$ are uncorrelated. Now, consider the two terms $\mathrm{Im}\{x_1x_1x_2^*x_0^*C_{1,1}\}$ and $\mathrm{Im}\{x_1x_2x_3^*x_0^*C_{1,2}\}$ in $\phi_0(t)$ and define

$$S_1 = x_1x_1x_2^*x_0^* \quad \text{and} \quad S_2 = x_1x_2x_3^*x_0^*.$$

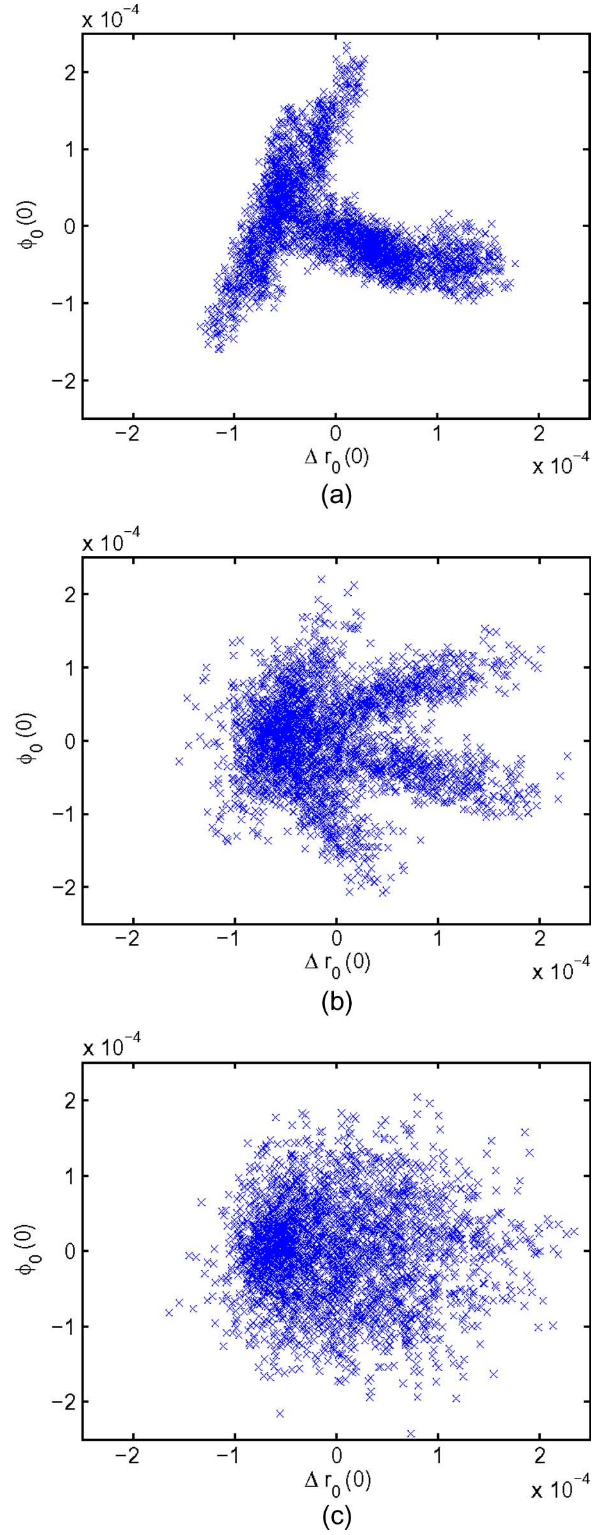

Fig. 2. Scatter plot of IFWM-induced phase and amplitude noise for a 40 GSym/s return-to-zero system after propagation through one span of standard single-mode fiber. The modulation formats are (a) BPSK, (b) QPSK, and (c) 8-PSK. The pulse shape is Gaussian with a signal power of 0 dBm.

The distribution of S_1 and S_2, $p_{S_1}(s_1)$ and $p_{S_2}(s_2)$ of S_2, are both uniform on the set $\mathcal{C} = \{e^{j2\pi/M}, e^{j4\pi/M}, \ldots, e^{j2\pi}\}$. At first glance, one might conclude that S_1 and S_2 are not independent of each other, as x_1, x_2, and x_0 are present in both S_1 and S_2. However

$$\begin{aligned} p_{S_1,S_2}(s_1,s_2) &= \sum_{x_0,x_1,x_2 \in \mathcal{C}^3} p_{S_1,S_2}(s_1,s_2|x_0,x_1,x_2)p_{x_0,x_1,x_2} \\ &\overset{(a)}{=} \sum_{x_0,x_1,x_2 \in \mathcal{C}^3} p_{S_1}(s_1|x_0,x_1,x_2) \\ &\qquad \times p_{S_2}(s_2|x_0,x_1,x_2)\left(\frac{1}{M^3}\right) \\ &= \sum_{x_0,x_1,x_2 \in \mathcal{C}^3} \left(\mathbf{1}_{\{s_1=x_0^* x_1^2, x_2^*\}}\right)\left(\frac{1}{M}\right)\left(\frac{1}{M^3}\right) \\ &= \frac{1}{M^2} = p_{S_1}(s_1)p_{S_2}(s_2) \end{aligned} \tag{18}$$

where p_{x_0,x_1,x_2} is the joint probability distribution of (x_0,x_1,x_2) and the conditional independence property in equality (a) results from the independence of x_i from x_j, $j \neq i$. The function $\mathbf{1}_{\{H\}}$ is an indicator function, which is 1 when the event H is true and 0 otherwise. Consequently, S_1 and S_2 are independent of each other. Using similar arguments, one can show that two random variables $x_a x_b x_c x_p$ and $x_a x_b x_c x_q$ are independent of each other as long as $p \neq q$. As a result, the terms in $\phi_0(t)$ are pairwise independent. Note that this property is a consequence of x_is' being phase-modulated with the phases obeying addition modulo 2π. With this insight, one can approximate the characteristic function $\Phi_{\phi_0(t)}(\omega)$ of $\phi_0(t)$ as

$$\Phi_{\phi_0(t)}(\omega) \approx \prod_{l,m\neq 0} \Phi_{l,m}(\omega) \tag{19}$$

where

$$\Phi_{l,m}(\omega) = \sum_{q=1}^{M} e^{j\omega \mathrm{Im}\left\{C_{l,m}e^{j2\pi q/M}\right\}} \tag{20}$$

is the characteristic function for the random variable $\mathrm{Im}\{x_l x_m x_{l+m}^* x_0^* C_{l,m}\}$. The corresponding probability distribution of $\phi_0(t)$ is given approximately by

$$p_{\phi_0(t)}(\theta) \approx \frac{1}{2\pi}\sum_{i=-\infty}^{\infty} g_i e^{ji\theta} \tag{21}$$

where $g_i = \Phi_{\phi_0(t)}(-i)$. Note that this is only an approximation, as the terms in $\phi_0(t)$ are not jointly independent. For example, for three of the terms $S_1 = x_1^2 x_2^* x_0^*$, $S_4 = x_1 x_{-1} x_0^{*2}$, and $S_5 = x_2 x_{-1} x_1^* x_0^*$ present in $\phi_0(t)$, one can show that $S_1 = S_4 S_5^*$ and hence they are actually deterministically related to each other. The empirical probability distribution of $\phi_0(0)$ obtained from simulation and the analytical approximation of (21) are shown in Fig. 3 for a terrestrial system consisting of 40 identical spans with parameters listed in Table I. The modulation format is QPSK and the inline amplifiers are assumed to be noiseless. The empirical distribution is obtained by a histogram plot of $\phi_0(0)$ for $-6 \leq l, m, l+m \leq 6$. A Gaussian fit to the empirical distribution is also shown as comparison. It can be seen that the analytical approximation is in good agreement with the empirical distribution. However, it cannot capture the asymmetry of the empirical distribution with respect to zero, as all the odd-order moments are zero in the analytical approximation (21).

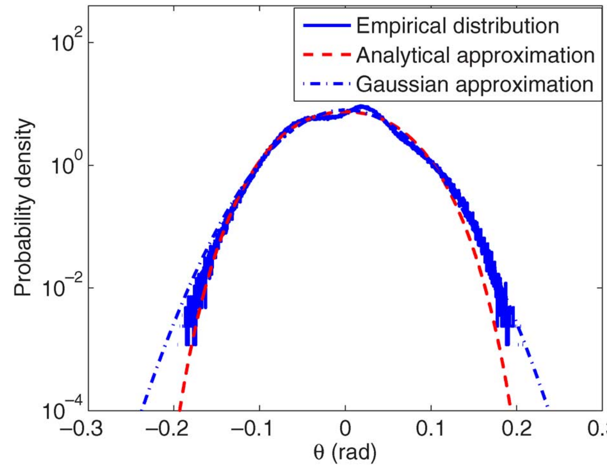

Fig. 3. Probability distribution of IFWM-induced phase noise for a QPSK system in a terrestrial link consisting of 40 identical spans. The mean nonlinear phase shift $\Phi_{\mathrm{NL}} = 0.86$ rad.

Using (21), one can analytically derive the phase-noise distribution in the presence of IFWM and receiver shot noise. Let $n \sim \mathcal{CN}(0, \sigma^2 I)$ represent the receiver noise (a combination of shot noise and thermal noise), which is assumed to be a complex circularly symmetric Gaussian random variable. If we denote $\rho = |u_0^{(l)}(0)|^2/2\sigma^2$ as the signal-to-noise ratio (SNR) at the center of the pulse, the phase noise Ψ induced by n is given by [26]

$$p_\Psi(\theta) = \frac{1}{2\pi}e^{-\rho} + \sqrt{\frac{\rho}{4\pi}}\cos(\theta)e^{-\rho\sin^2\theta}\mathrm{erfc}(-\sqrt{\rho}\cos\theta) \tag{22}$$

or

$$p_\Psi(\theta) = \frac{1}{2\pi}\sum_i h_i e^{ji\theta} \tag{23}$$

where

$$h_i = \frac{\sqrt{\pi\rho}}{2}e^{-\rho/2}\left[I_{\frac{i-1}{2}}\left(\frac{\rho}{2}\right) + I_{\frac{i+1}{2}}\left(\frac{\rho}{2}\right)\right]. \tag{24}$$

As ϕ_0 and Ψ are independent of each other, the characteristic function of the overall phase noise $\phi_0 + \Psi$ is simply the product of the two individual characteristic functions. Now, given a probability distribution, define $Q(\theta)$ as its two-sided tail probability. In this case, for the analytical approximation of ϕ_0 corrupted by Ψ

$$\begin{aligned} Q(\theta) &= 1 - \int_{-\theta}^{\theta}\sum_i g_i h_i e^{ji\theta'}d\theta' \\ &= 1 - \sum_i \frac{g_i h_i}{2}\sin(i\theta). \end{aligned} \tag{25}$$

A plot of $Q(\theta)$ versus θ is shown in Fig. 4. From the figure, the tail probability corresponding to the analytical approximation matches well with that from the empirical distribution. On

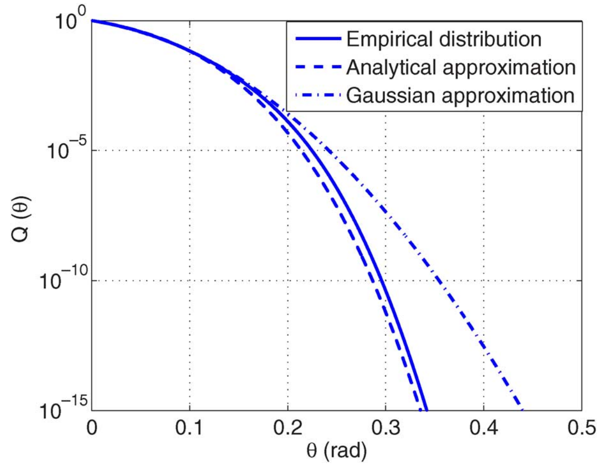

Fig. 4. $Q(\theta)$ for a QPSK system in presence of IFWM-induced phase noise and receiver noise in a terrestrial link consisting of 40 identical spans. The mean nonlinear phase shift $\Phi_{\rm NL} = 0.86$ rad and the SNR is 30 dB.

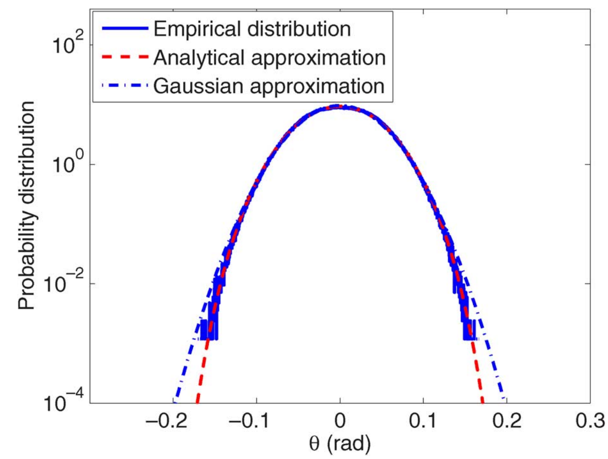

Fig. 5. Probability distribution of IFWM-induced differential phase noise for a DQPSK system in a terrestrial link consisting of 40 identical spans. The mean nonlinear phase shift $\Phi_{\rm NL} = 0.86$ rad.

the other hand, the Gaussian fit significantly overestimates the tail error probability. For M-ary PSK systems limited by phase noise, the symbol error rate will be $Q(\pi/M)$.

For differentially modulated systems, similar methods, with some modifications, can be used to approximate the probability distribution of the IFWM-induced differential phase noise. It is given by

$$\begin{aligned}\theta_d &= \phi_0 - \phi_1 \\ &= \sum_{l,m\neq 0} \mathrm{Im}\left\{x_l x_m x_{l+m}^* x_0^* C_{l,m}\right\} \\ &\quad - \sum_{p,q\neq 1} \mathrm{Im}\left\{x_p x_q x_{p+q-1}^* x_1^* C_{p-1,q-1}\right\}. \end{aligned} \tag{26}$$

From the discussion of $R_\phi(k,t)$ in the previous section, it can be shown that some terms from $\phi_0(t)$ and $\phi_1(t)$ are correlated (for example, $\mathrm{Im}\{x_1 x_1 x_2^* x_0^* C_{1,1}\}$ from $\phi_0(t)$ and $\mathrm{Im}\{x_0 x_2 x_1^* x_1^* C_{-1,1}\}$ from $\phi_1(t)$). As a result, if we assume the terms in $\phi_0(t) - \phi_1(t)$ are independent from each other, the approximation to the probability distribution of θ_d might not be accurate. In this case, we can group terms that are correlated and treat them collectively as a single random variable. The new terms in $\phi_0(t) - \phi_1(t)$ will be pairwise independent again. The empirical probability distribution of $\phi_0(t) - \phi_1(t)$ and the corresponding $Q(\theta)$ are shown in Figs. 5 and 6, respectively, with their analytical and Gaussian approximations for a differential QPSK (DQPSK) system. From the figures, it can be seen that the analytical approximations to the probability distribution, as well as the tail probability, are almost exact. In contrast, the Gaussian approximation again significantly overestimates $Q(\theta)$.

IV. Exploiting the Statistics of IFWM Phase Noise

The analytical insights into the correlation and the probability distribution of IFWM-induced phase noise enable signal processing to improve system performance. For example, one can exploit the knowledge of $R_\phi(k,t)$ by implementing an optimal linear phase-noise predictor. In particular, let the phase of the kth received symbol be

$$\theta_k = x_k + \phi_k$$

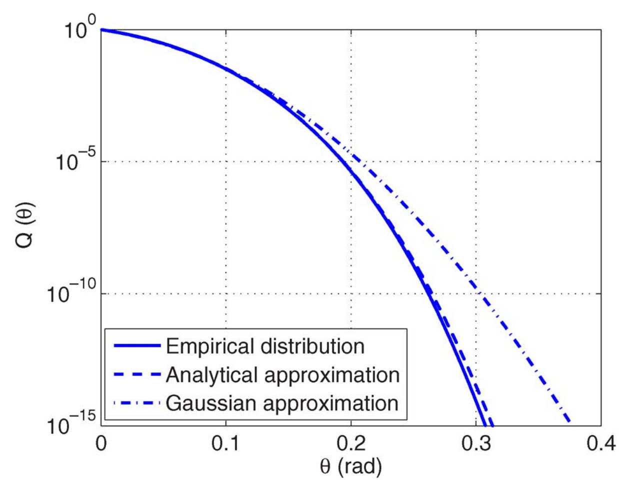

Fig. 6. $Q(\theta)$ for a DQPSK system in presence of IFWM-induced differential phase noise and receiver noise in a terrestrial link consisting of 40 identical spans. The mean nonlinear phase shift $\Phi_{\rm NL} = 0.86$ rad and the SNR is 30 dB.

assuming sampling at pulse peak and neglecting the effect of optical and electrical filtering. If the symbols $x_0, x_1, \ldots, x_{k-1}$ are correctly detected, $\phi_0, \phi_1, \ldots, \phi_{k-1}$ can be calculated. One can then pass the ϕs to a finite impulse response filter $F = [f(1) \ \ldots \ f(k-1)]$ such that the decision variable for the kth symbol is

$$\theta_k' = x_k + \phi_k - \sum_{i=1} f(i)\phi_{k-i}.$$

The filter coefficients that minimize the phase-noise variance of θ_k' are given by

$$F = T^{-1}\begin{bmatrix} R_\phi(1,0) \\ R_\phi(2,0) \\ \vdots \\ R_\phi(K,0)\end{bmatrix} \tag{27}$$

where $T_{i,j} = R_\phi(|i-j|, 0)$. It can be shown that when IFWM-induced phase noise dominates over other phase noises, a 1.8-dB improvement can be obtained using the noise predictor. In a realistic system, however, ASE noise from inline amplifiers and nonlinear phase noise increases the overall phase-noise variance. A comparison of $R_\phi(k,0)$ from (12) with that in the pres-

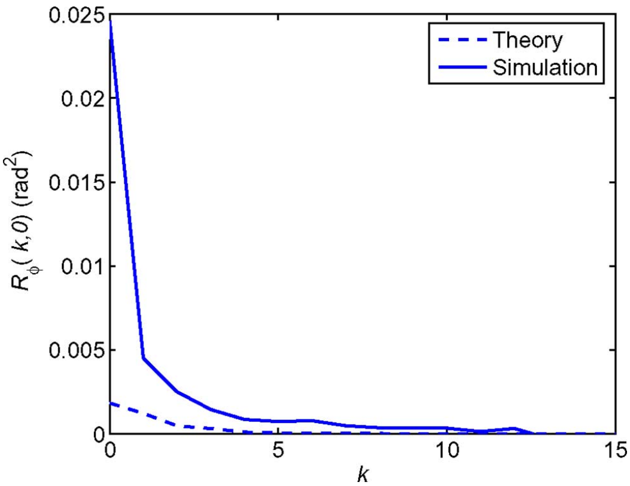

Fig. 7. Comparison of (12) with phase-noise correlation for a terrestrial system consisting of 40 spans in presence of IFWM phase noise, ASE noise, and nonlinear phase noise. The launched signal power is -1 dBm and the noise figure of the amplifiers is 4.5 dB.

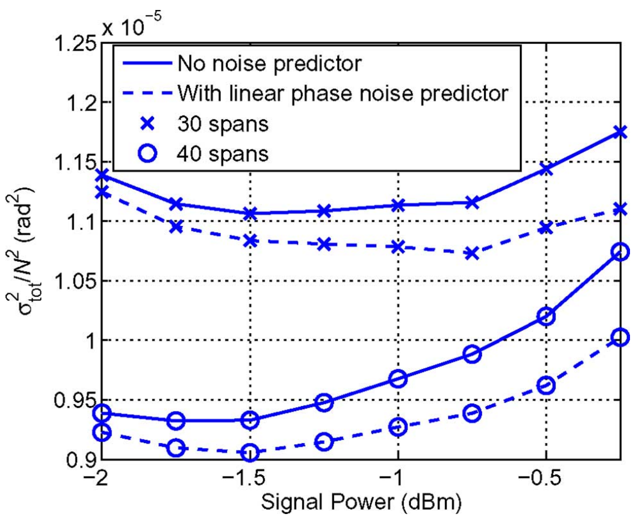

Fig. 8. Phase-noise variance for a terrestrial system in presence of IFWM-induced phase noise, ASE noise, and nonlinear phase noise with optimal linear phase-noise prediction.

ence of IFWM-induced phase noise, ASE noise, and nonlinear phase noise is shown in Fig. 7 for a system with 40 identical spans and two-stage amplification per span. The figure clearly illustrates that the presence of ASE noise and nonlinear phase noise significantly reduces the correlation of the overall phase noise across symbols. In deriving analytically the phase-noise autocorrelation function that is used to derive F, we use a modified correlation function $R'_\phi(k,0)$ with

$$R'_\phi(k,0) = \begin{cases} R_\phi(0,0) + N\sigma^2_{\mathrm{ASE}}, & k=0 \\ R_\phi(k,0), & \text{otherwise} \end{cases} \quad (28)$$

where N is the number of spans in the link and σ^2_{ASE} is the phase-noise variance induced by ASE noise in one span. We implement the optimal linear phase-noise predictor based on $R'_\phi(k,0)$, and the resulting total phase-noise variance σ^2_{tot} is shown in Fig. 8 as a function of signal power. The variance values are normalized by N^2. The simulation used 10^5 trial propagations to estimate the phase-noise variance for each signal power level. From the figure, we see that an improvement of about 0.8 dB for a system of 40 spans and about 1.2 dB for 30 spans is obtained.

Finally, if we denote the power spectral density of ϕ as

$$S_{\phi\phi}(z) = \mathcal{F}\{R_\phi(k,0)\} = \sigma^2\Gamma(z)\Gamma(z^*) \quad (29)$$

one can decorrelate the channel by passing the sequence of received phases through a whitening filter with z-transform $1/\Gamma(z)$. The output phases are then given by

$$\theta'_k = \sum_{i=0} W_i x_{k-i} + \sum_{i=0} W_i \phi_{k-i} = \sum_{i=0} W_i x_{k-i} + \phi'_k \quad (30)$$

where the W_is are the weights of the whitening filter. In this case, the channel becomes an ISI channel with the ϕ'_ks uncorrelated. If they are also close to being independent of each other, one can then implement maximum-likelihood sequence detection (MLSD). The distribution of ϕ'_k can be obtained in a way similar to that in deriving the distribution of $\phi_0 - \phi_1$ for DPSK systems. The amount of performance improvement obtained using MLSD depends on the relative importance of IFWM-induced phase noise compared to other noises and the accuracy of the channel statistics used in the detection process.

V. Conclusion

In this paper, we studied the statistics of intrachannel four-wave mixing in phase-modulated communication systems. We studied the correlation of the IFWM-induced phase and amplitude noises and showed that they are correlated with each other for BPSK but uncorrelated for M-ary PSK, $M > 2$. In addition, we presented methods to approximate the probability distribution of IFWM-induced phase noise for PSK and DPSK systems. We verified the analytical predictions through simulation. We studied the performance of an optimal linear phase-noise prediction filter derived from the IFWM-induced phase-noise autocorrelation. We obtained an improvement of 1.8 dB when IFWM-induced phase noise dominates and 0.8–1.2 dB for typical terrestrial links in the presence of ASE noise from inline amplifiers and nonlinear phase noise. Further applications of the knowledge of IFWM statistics to maximum-likelihood sequence detection and multichip detection of DPSK systems will be investigated in the future.

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